LEONHARD EULER *and the* BERNOULLIS

M. B. W. Tent

Leonhard Euler and the Bernoullis

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Mathematicians from Basel

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To our friends,
Sabine and Christian Koch

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Preface

These mathematicians, who lived between 1650 and 1800, all grew up in Basel, Switzerland. The first two—Jacob and Johann Bernoulli—were important Bernoulli mathematicians who made their careers mainly in Basel. Jacob's name is sometimes given as James in English or Jacques in French, and Johann's name is sometimes given as John in English or Jean in French. Johann's son Daniel, the third great mathematical Bernoulli, spent some years early in his career first in Venice and then in St. Petersburg but returned to Basel as soon as he was able to arrange it. All the other Bernoullis made their careers in Switzerland whenever possible. Only Euler (pronounced "oiler"), who made his career in St. Petersburg and Berlin, chose not to return to Basel. Since the Bernoullis were all related and tended to use the same first names over and over, their names can be confusing, but I hope the family trees in the text will help the reader keep them straight. There is only one Euler who made a career as a mathematician, but he was as important to the development of mathematics as all the Bernoullis taken together. It is unfortunate that most Americans, unless they are crossword puzzle enthusiasts, have never even heard the name Euler. And for that matter, most Americans have heard only of Daniel Bernoulli even though his father Johann and his uncle Jacob were probably equally important.

As I assembled this story, I was disturbed by the minor role played by the women. If the Bernoullis had what might be called the "math gene," surely that was present in the females as well as the males. I assume that the mothers were significant in the upbringing of both boys and girls, although there is also little indication of that influence in the historical record. I think it is likely that Daniel Bernoulli's older sister Anna Catharina was at least partially involved in Daniel and Nicolaus' discussions of mathematics when they were growing up, but that is conjecture on my part. The Bernoulli girls, like other girls at that time, were probably barred from serious education and from later life in the academic world simply because of their gender and the time that they lived. That is regrettable.

The information available on Euler and the Bernoullis is spotty, and in order to make a coherent story I had to fabricate some of the details of their lives and the dialogues that portray their interactions. In general I have tried to convey the interactions of the families and the mathematicians in a way that is compatible with the available records, but there is certainly an element of fiction throughout this work. The quotations from letters are only loose translations, but I have tried to convey both the gist and the mood of the letters. They were written in German, Latin, and French, and I have not made a note of the languages except in one letter that Daniel Bernoulli wrote to Euler, in which he switched repeatedly from one language to another. It is interesting that the correspondents generally preserved the grammar of the disparate languages as they switched from language to language. Their formal letters were all written exclusively in Latin, the language of the scientific community of Europe at the time, while many of their casual letters were written in one or more languages.

Many of the "brilliant but bickering Bernoullis," as William Dunham called them, were indeed cantankerous, particularly when it came to guiding their sons into their careers. For some reason, each succeeding generation apparently tried to force sons into business, law, or medicine rather than mathematics. I have indicated that attitude in part by showing the Bernoulli patriarchs often responding to their sons with a resounding "no!" Leonhard Euler and his father, by contrast, were apparently always kind and supportive as they brought up their children, often responding to the younger generation with a pleasant "yes." It seems to me that that distinction fits with the record, although we have no indication of their use of yes and no.

Another trait the Bernoullis share is that, no matter how cantankerous they were, beginning with Johann they all respected and genuinely liked Euler. That is particularly touching when we consider the contrast between the way the first mathematical Johann Bernoulli treated his son Daniel to the way in which he treated his protégé Euler. Apparently Daniel didn't resent Euler, showing a serenity almost unheard of in a slighted son.

There seems to be general agreement among mathematicians that Euler was one of the four greatest mathematicians of all time, sharing that distinction with Archimedes, Newton, and Gauss. Some have suggested that the whole Bernoulli family should constitute the fifth great mathematician. Among them, these Basel mathematicians had a major impact on the development of mathematics, as well as physics, astronomy, and many other related fields. The two families are certainly responsible for the presentation of Leibniz's calculus to the world, and that alone binds them together.

Since the world may never again see a mathematical clan like Euler and the Bernoullis, it is important that we recognize them for their phenomenal accomplishments and contributions to mathematics. The citizens of Basel didn't ask for a dynasty of mathematicians, but that is what they got. The rest of us can benefit from them as well, but only if we know their story.

Acknowledgments

I want to begin by thanking two remarkable young women who helped me generously in the preparation of this manuscript. Sulamith Gehr, an archivist in Basel, Switzerland, helped me repeatedly, often devoting her precious personal time to tracking down sources for me and later reading my entire manuscript carefully and providing detailed corrections. As we corresponded over the last 18 months, she has never complained about locating the source that I needed and scanning it for me. It is safe to say that without her help this work would be far less accurate and complete than it is. Thank you, Sulamith.

The second young woman whom I want to thank is my daughter, Virginia Tent. While working full time, she managed to find time during her daily subway commute to read the entire manuscript—some parts of it multiple times. Her suggestions showed a real feel for what I was trying to accomplish. On more than one occasion, she urged me to put in more human feeling or to flesh out certain scenes. Her help is particularly memorable on the section where Jacob Bernoulli describes his commitment to mathematics to his reluctant father. The entire book reads better because of Virginia's attentions. Thank you, Virginia.

Next I would like to recognize my two photographers. Lizanne Gray traveled with me to Berlin and Basel in the fall of 2007, taking many, many pictures, both of what I asked her to and what she thought would be appropriate. The result is a wonderful collection of photos that portray many aspects of this story. The lion's share of the photos in this book are Lizanne's work. In addition to Lizanne, my sister-in-law Rosemary K.M. Wyman took two of the photos when I was visiting in Maine. I asked her if she could get a picture of the water flowing under the bridge in the Bagaduce River in Maine and of a snail shell that Virginia Tent found on the shore. Both those photos are masterful. Thank you, Lizanne and Rosemary, for your artistic eyes and technical skill.

I would like to thank my brother, David Wyman, for his help on the work of Daniel Bernoulli. My background in physics is sketchy, but with his knowledge of boats and moving water, David was able to correct my descriptions of navigation and the Bernoulli Principle. It was important that I get those sections right. Thank you, David.

Amanda Galpin, a fine graphic artist, was willing to learn enough about the cycloid to draw its path, depicting a marked wheel as it rolls along a straight path. It is nothing she had ever worried about before, but she approached the challenge directly and quickly, producing what I think is a masterful drawing. Thank you, Amanda.

I needed occasional help in translating some of my sources as well. Although I speak German and French and theoretically read Latin, producing a good English translation of those languages was sometimes beyond my skill level. Jeanne Classé and Jake Linder, teachers of French and Latin respectively at the Altamont School, were repeatedly helpful in fine-tuning my translations. In addition, I should once again thank my daughter Virginia for her help in translating German and French documents. I say to you three, *gratias vobis ago*, *merci beaucoup*, and *danke schön*!

I would like to thank two other archivists in Basel. Dr. Fritz Nagel spent several hours showing Lizanne Gray and me where we needed to go on our walking tour of Basel as we photographed the Bernoullis' environs, and he was most helpful in setting me up for my research in the Bernoulli Archive. Martin Mattmüller at the Euler Archive in Basel was most accommodating as he provided me with sources from his archive as well as a charming paper weight featuring the Leonhard Euler stamp. I particularly appreciate Herr Mattmüller's willingness to send me scans of some documents that I needed to access from Birmingham. Herr Mattmüller's translation of Jacob Bernoulli's poem about infinity is the best that I have found anywhere. Both these archivists provided important material and background information for me. Thank you Dr. Nagel and Herr Mattmüller!

The staff at the Prussian Academy of Sciences in Berlin were most accommodating in providing me with documents and material, and allowing Lizanne Gray to photograph some of their documents. We were particularly charmed with the 1753 almanac, which she photographed in detail. Thank you to the archive staff for their generous help!

Ellen Griffin Shade and Jonathan Newman at the Avondale Branch of the Birmingham Public Library were able several times to help me locate reference materials through their library, often searching for what must have seemed truly bizarre to them. Thank you!

Two of my friends read the manuscript intelligently, giving me some excellent feedback as I revised sections. Mia Cather wanted dates and ages of the characters involved—an excellent suggestion!—and she was also extremely helpful in providing information on her hometown, Groningen, Holland, where Johann Bernoulli served as professor for ten years. Naomi Buklad studied my prose carefully and made several cogent points. Thank you, Mia and Naomi!

At A K Peters, Klaus Peters was supportive and creative in his reactions to my writing. Klaus had a clear vision for this book even when it was in the early stages, and I believe he was right. I sincerely appreciate his comments and suggestions. Charlotte Henderson has always been patient with me, helping me see what I needed to see and providing technical help when I needed it. This book would never have been born without Klaus' and Charlotte's help. Also through A K Peters, Erika Gautschi's copyediting was perceptive and precise. Because she caught several critical errors that I had made in addition to her general editing, this is a far better book than it would have been without her work. I thank you all!

Finally, I would like to thank my husband, James F. Tent. As a professor of German history, he was able to fill in the background that I needed as I wrote—for example, about the persecution of the Huguenots and the role of Peter the Great's Russia in the Europe of the time. Jim also read the manuscript and provided me with important reactions to several sections as I was revising it. I also greatly appreciate that fact that he has supported me in my retirement from teaching, encouraged me, and gone with me in travels to Europe whenever his academic calendar allowed it. Thank you, Jim, as always for your understanding and encouragement.

There are undoubtedly others whom I should mention here, and I apologize to anyone I have omitted. However, I should say that any errors in this book are mine—those who assisted me were wonderful, but I am the one who is responsible for the resulting work.

Figure Credits

Unless otherwise noted below, photographs are by Lizanne Gray and illustrations are by the author.

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1
The Bernoullis as Huguenots

"Peter, won't you take some cheese and pass it on?" Francina Bernoulli said to her oldest son as they sat at breakfast one morning in the bustling city of Antwerp in the Spanish Netherlands in the year 1567. "The rest of us are hungry too."

"Oh, Mother!" Peter said, passing the cheese board to his father and taking his first bite. "This is the best cheese!"

"Yes," his mother said, "it's gouda, and it's very fresh. I know you like it best when it is still young, as the cheese maker describes it."

"It's so good!" Peter said enthusiastically.

Francina turned to her husband Jacob, "Did you ever see anyone eat so much?"

"He's a growing boy!" Jacob said. "I remember how hungry I was at his age. By the way, I'll be meeting later today with Justus de Boer. He and I have been exploring working together on shipments of some exotic spices from India. I think it's very exciting."

"I like Justus" Francina said. "I can't think of anyone better to work with."

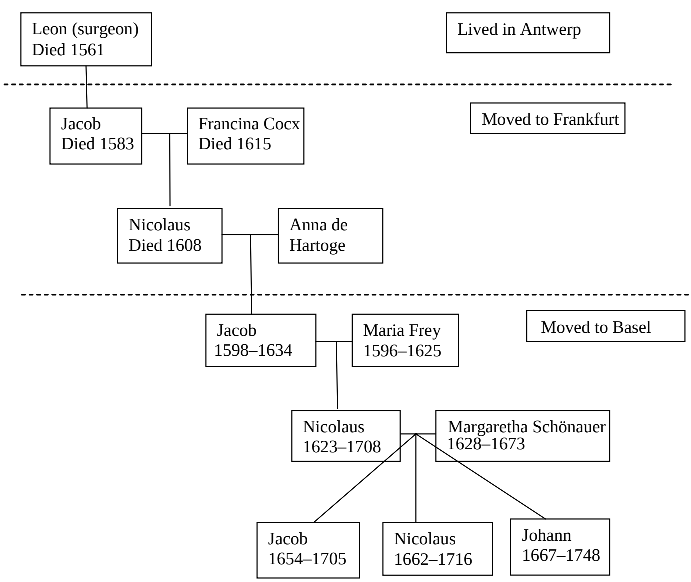
"No," Jacob said, "I can't either. Not only is he honest and hard-working—he's also smart. You can't ask for more than that in a friend and colleague!"

"Jacob!" Francina Berrnoulli called to her husband as he returned home from work that evening. "Did you hear about Jan Suratt? They burned him alive! Everyone says it is because he refused to acknowledge the Pope! They say the crowd screamed that he was a heretic—that they shouted over and over that he deserved to die!"

"Yes, I heard," Jacob Bernoulli said grimly. "They also burned Justus DeBoer at the stake last night." Jacob sat down at the table and sadly rested his head in his hands.

"Justus? Your friend Justus?" Francina gasped. She quietly put her hand on Jacob's shoulder as together they contemplated the horror of Justus' fate.

"Yes, I know," Jacob said. "Think what my father would have said!" Jacob's father Leon, a devout Protestant, had been a pharmacist and surgeon in Antwerp. He had been one of the leaders in



Bernoulli family tree, Antwerp to Basel, 1550–1750.

that exciting port city that was then the thriving center of the Spanish Netherlands' international trade. Leon had been committed to helping his fellow man in every way that he could, and as a surgeon he did what he could to ease the suffering of all people. Among his patients were Protestants and Catholics, Jews and Gentiles, Hollanders and foreigners, and to him the patient's background or religious preferences were irrelevant. He lived the Hippocratic oath: do as much good as possible, but at the very least do no harm. How could the predominantly Protestant city of Antwerp only one generation later have become the scene of deliberate, cruel torture of some of its most respected citizens?

"Jacob, I'm afraid," Francina admitted as she quietly sat down beside him. "The authorities know that we are Protestants, don't they?"

"I'm sure they do," Jacob said. "The Spanish Duke of Alba has made it his business to know such things. He calls us infidels because we have left the Catholic Church."

"Oh, Jacob," Francina said, tears welling up in her eyes, "Do you think we need to leave Antwerp?"

"Yes, I think we should, and I fear we should do it quickly," Jacob said as he shuddered, looking sorrowfully at his wife. Then he continued, "How could they have done this to him? Justus was no threat to them. He wasn't plotting a revolution. He was a good man who always tried to do what was best. He was exactly the kind of man that a civilized mercantile city like Antwerp needs. Why did they care where he chose to go to church—how he chose to worship God? Those are private choices. All people should be able to make those choices for themselves. Oh, dear. Without him and people like him, this center of international trade is nothing. How could they have killed him? It's an abomination!"

"I know," Francina said, taking Jacob by the hand. "He was a fine man."

"Yes, he was," Jacob said. "He was one of the best." Then taking control of his emotions, he continued, "All right, here is what I think we should do: let's go to Frankfurt on the Main River. From what I hear, Protestants are thriving there. It is well known that the Spaniards have no influence in that Free Imperial City of the Holy Roman Empire, so I should be able to continue my business there without fear of persecution. I think it will be best to limit my business there to medicinal spices and drugs, since the diamond trade is best handled from the seaport here. Fortunately for us, establishing the spice trade in Frankfurt is the logical next step in international trade."

"That sounds good, Jacob," Francina said.

"I made inquiries today," Jacob continued, "and I learned that there will be a boat going up the Rhine River from Rotterdam a week from today. I think we should be on it. A carriage would be faster, but because a boat will allow us to take as much as we need, it seems like the best way to go. I spoke today with several of our fellow Protestants, and we agreed that it is best for us to make the move first. You and I will go to Frankfurt with our children first. Because the others are weaker financially, they will have to stretch to make the move. I think it is our responsibility to pave the way for them, and we can do that. If they are cautious and quiet, I hope they won't get caught like Justus and Jan. Once we get established, we can prepare for the others to come as well."

"Yes, Jacob," Francina said. "I think you are right. Your successful business and the money I inherited from my father have set us up well to do this."

"So we will need to leave Antwerp on Monday," Jacob said. "The trip up the river will be very slow—pulling a big boat up the mighty Rhine River is a difficult task—but horses are strong, and they can do it. I hope we will be able to slip away without attracting any notice from the authorities. I'll reserve places for all six of us on the boat."

"I'll start packing at once," Francina said. "Today is Wednesday—we don't have much time! I don't like it, but you are right: we don't have a choice."

"Well, the only alternative would be to convert to Catholicism," said Jacob, "and after what happened yesterday I cannot do that. I am unwilling to submit to the authority of the Pope ever again."

"No, neither of us can do that," Francina agreed. "I will spend tomorrow and the next day sewing gold pieces into the seams of your other shirt and my petticoats. Maybe I can do that to Peter's shirt as well. I think he's old enough for that, don't you? Gold is probably the most portable resource we can take and we have quite a lot, but I will also pack as many clothes for the children as I can. Oh, dear, Jacob! I don't like this at all."

"I don't either, and I agree that sewing gold pieces into Peter's shirt is a good idea. I have some perfect diamonds at the office that you could sew into our clothes as well," Jacob said. "I'll bring them home with me tomorrow. They aren't as heavy as gold, and for their weight they are quite a lot more valuable."

"That's a good idea," Francina said. "Jacob, I'm glad you see it the way I do. I was afraid you might want to stay here and fight. It is appalling that the Duke of Alba is doing this to us!"

"Yes, it is," Jacob agreed, "and perhaps if I were alone I might risk staying here and fighting, but it is unfair to put you and the children in such danger, and yesterday's events prove that the danger is very real. Once we get to Frankfurt, we should be able to prepare the way for all our like-minded friends to come join us, God willing. I pray that they will survive until then."

The Bernoulli family's move was timely. They were able to provide leadership for the Antwerp Protestants in Frankfurt, helping the entire group thrive in their adopted city. Only five years later in 1572, at least 10,000 French Huguenots [Protestants] died in the massacre in France on St. Bartholomew's Day, signaling the beginning of outright war between Catholics and Protestants in Europe. Four years after that in 1576, Antwerp, the primarily Protestant city where the

Bernoullis had lived, was the scene of another cruel slaughter of Huguenots. As many as 8,000 supposed heretics were killed in Antwerp by the troops of the Spanish Duke of Alba on the first day alone, and that included men, women, and even children! After three days, there were no more Huguenots anywhere in Antwerp—they were either dead, or they had escaped with only the clothes they were wearing because of what came to be called the "Spanish Fury." Some had drowned in the river Scheldt after jumping in a final act of desperation. The part of the Netherlands that was under Spanish rule had become a death trap for Protestants, but by now the Bernoullis and their fellow Protestant refugees from Antwerp were thriving in the Free Imperial City of Frankfurt, far from the violence in their native city.

2
The Bernoulli Family in Frankfurt and Then Basel

Frankfurt welcomed the Bernoulli family, and Jacob's business—importing spices from East Asia—was as successful as he had predicted. The family easily made the switch from the Dutch language to German as they adjusted to life in the Rhine-Main region. Jacob and Francina had a total of 17 children although many of them, succumbing to the common diseases of the time, didn't survive beyond their fifth birthdays. By 1570, only three years after his flight from Antwerp, Jacob had become a Frankfurt city councilor because of his impressive success as a businessman. He enjoyed widespread respect in his adopted city. Although at this point a talent in mathematics had not yet been recognized among the Bernoullis, Jacob was clearly able to keep his accounts straight and to make a profit consistently.

Jacob's son Nicolaus continued the family spice business in Frankfurt until 1592, when he moved to the Protestant city of Amsterdam in Holland for a time with his wife Anna. Although he might have wished to return to the family roots in Antwerp, that was not an option. Following the "Spanish Fury," Antwerp had become the most Catholic city in northern Europe, with no tolerance for wayward Protestants. A few years later, Nicolaus returned to Frankfurt to continue the family business there.

In 1620, Nicolaus' son Jacob (grandson of Jacob and Francina who had fled from Antwerp fearing their persecution as Huguenots) decided to move farther up the Rhine River to Basel in what was then called the Helvetian Confederation—what is now Switzerland. With this move, he was removing his family and business from the threats of the emerging Thirty Years War, which ravaged central Europe from 1618 until 1648. By 1622 Jacob, already a well-respected businessman in Basel, was appointed city councilor, probably with some help from the family of his new wife Maria Frey, who was the daughter of a prominent banker and Basel city magistrate. In Basel, the Bernoulli family business in spices continued to prosper.

Jacob and Maria's second son Nicolaus married Margarethe Schönauer, the daughter of a successful pharmacist in Basel, and two of their sons—Jacob and Johann—became the first mathematical Bernoullis, four generations after the family's flight from Antwerp. The mathematical dynasty of the Bernoullis would continue to produce respected mathematicians at an astonishing rate for more than 100 years.

Ever since, mathematicians have argued about whether the Bernoullis had the "math gene"—whatever that might be—or whether each successive generation was somehow brought up to have a passion for mathematics despite their fathers' wishes. Certainly mathematics was never openly encouraged in the family. The "nature or nurture" question in the Bernoulli family is still unresolved, but no one can deny that the family produced at least eight truly great mathematicians within three generations, beginning with the two brothers Jacob and Johann.

In 1668, with the family business now well established in Basel, Nicolaus decided that his very intelligent oldest son—14-year-old Jacob—needn't follow the harried career in business of his father and grandfathers before him.

"Jacob," Nicolaus said to his oldest son one evening, "I have decided that you may be better suited to an intellectual life than to a life in the business world."

"Really?" Jacob asked. "Do you mean that I might study at the university?"

"Yes, I think that would be wise," Nicolaus said. "I've noticed that you are not a fast talker—that you seem to think carefully before you speak. I am almost tempted to say that you seem to have more of a brooding personality—you often seem meditative and deep in thought. What would you think about pursuing a career in the Church?"

"I think I might like that," Jacob agreed. "I must say that a career in business doesn't particularly appeal to me."

"So, I believe what you should do is to study philosophy first," Nicolaus explained, "and then you would move on to the serious study of theology."

"Yes, I like that idea," Jacob said. "In fact, that is what my friend Hans will be doing."

"I'm glad to hear that," Nicolaus said. "You are making me very happy, my boy!"

As directed by his father, Jacob studied philosophy at the university in Basel, and then, after completing his master's degree, he began the study of theology. However, without his father's knowledge, Jacob quietly elected to learn mathematics as well. Since his father expected his children to follow his directions fully, he was furious when he found out.

"Jacob, what is that book you are reading?" his father asked suspiciously one evening.

"It's mathematics, Father," Jacob cheerfully replied. "Most of my reading is in philosophy, but I believe a sprinkling of mathematics is a good balance. Don't you think so?"

"Mathematics?" his father asked. "No! What use could you have for that? Remember, we have reached the point where you can be more than just a businessman. Philosophy is far more important. Since you are a good student, my plan for your career is appropriate."

"But Father," Jacob protested, "You have said that I need to be an educated person, and you must admit that mathematics is certainly part of a broad education. Nothing is as purely abstract as mathematics—not philosophy or even theology."

"No!" his father exploded. "You already know enough mathematics. You learned plenty of that while you were in school, and there is really nothing more to it. You can already do all the reckoning you will ever need to do."

"But Father," Jacob persisted, "I think you don't really understand what mathematics is. It is far more than simple arithmetic. You wanted me to study philosophy, and I have been happy to do that. Plato, one of the greatest philosophers of all time, saw mathematics as the vehicle that draws the soul toward truth. In *The Republic*, his major work in philosophy, Plato argues that the study of mathematics (and by that he means pure mathematics—not just arithmetic) allows one's mind to reach the most ideal truths. He sees mathematics as the perfect vehicle for disciplining the mind. See? My study of philosophy requires me to pursue mathematics, an integral part of that noble subject. I am simply following your directions intelligently."

"Nonsense!"

"That is where you are wrong, Father," Jacob boldly corrected his father. "I have learned that there are some very exciting ideas to be found in pure mathematics, and I have only begun to study them. I would like to understand them all. You wouldn't believe how fascinating it is!"

"That is not what I sent you to the university to study," his father said. "Put that book away!" and with that his father lit a fresh candle, picked it up resolutely, and stormed out of the room.

The book Jacob was studying, which had been published more than 100 years earlier in 1544, was Stiefel's revised version of Christoff Rudolph's *Coss*, an algebra textbook originally published in 1525. The mathematics professor at Basel University had recommended it to Jacob when Jacob asked him what he should read in order to


Die Coss
Christoffs Rudolffs
Mit schönen Exempeln der Coss
Durch
Michael Stifel
Gebessert vnd sehr gemehrt.
Den Innhalt des gantzen Buchs
such nach der Vorred.
Zu Königsperg in Preussen
Gedrückt/ durch Alexandrum
Lutomyslensem im jar
1553.

Rudolph's Coss.

follow Plato's advice and learn more about mathematics. It was the first serious textbook of mathematics beyond basic arithmetic that was available in German, the Bernoullis' language. It presented algebra without the benefit of letters for variables—instead Rudolph used a word (such as the Latin word *facit* [makes] or the German word *gibt* [gives] for our symbol =) or sometimes an abbreviation for a word, to stand for an operation or for the unknown.

Although the mathematics in the *Coss* looks nothing like modern algebra, the *Coss* allowed a student to approach some of the problems found in algebra today, and it was the only way that anyone knew to do algebra at the time. The title *Coss* comes from the Italian word *cosa* [thing], a word that Rudolph sometimes used as his variable. At this time algebraists were often called *cossists*. Jacob had to study the *Coss* seriously if he wanted to pursue his study of mathematics—which he clearly was determined to do.

3
Jacob Makes His First Steps in the Study of Mathematics

Beginning on page 6 of the *Coss*, Jacob found an explanation of series—progressions. He carefully talked himself through the explanation: "All right. First Rudolph presents arithmetic series, in which I should always add the same amount—the common difference—as I move from one term to the next. His first series is the first seven counting numbers—1, 2, 3, 4, 5, 6, 7—in this case I simply add one for each new term. That's easy.

"Now, Rudolph is showing me a trick to find the sum of this series. He says all I have to do is to add the first and last terms—that would be 7 + 1 = 8—and then multiply the result by the fraction 7/2 to find the total. Now, where did he get that fraction? He must have used seven because there are seven terms, but what about the two? Oh, silly me! Of course! When I add 7 + 1, I am adding a pair of numbers. In fact there are 3 1/2 or 7/2 pairs of numbers in this series, and each pair must add up to a total of eight. That explains it. I just multiply by the number of pairs. When I multiply 7/2 times eight, that would give me 7/2 · 8 = 28, and yes, if I add 1 + 2 + 3 + 4 + 5 + 6 + 7, I get 28. That's good. I like it. Does Rudolph give me another arithmetic series?" Jacob asked himself.

"Yes, the next series is 6, 9, 12, 15," Jacob read. "Now first, I need to be sure that this is an arithmetic series. I see it. There is a common difference of three: 6 + 3 = 9, 9 + 3 = 12, 12 + 3 = 15. That's right. There are four numbers in the series, and, when I add the first and last terms, 6 + 15 = 21. This time I should multiply the sum of 21 by the fraction 4/2, since there are four numbers in the series, and so there must be 4/2 pairs. Since 4/2 = 2, the total must be 21 · 2 = 42. That's a good trick!

"Here's another series: 2, 4, 6, 8, 10, 12, 14. The difference between terms is two, there are seven terms, and the sum of 2 + 14 (the first and last terms) is 16, so I should multiply 16 · 7/2 = 56. Yes, that's what Rudolph gets, and when I add the terms, that's what I get too.

"Now I believe I understand arithmetic series," Jacob said to himself, "but now Rudolph is moving on to geometric series. I know that with an arithmetic series, there is a common difference between terms, but what about a geometric series? Aha! Instead of adding the same amount from term to term, this time I have to multiply by the same amount. So in the first geometric series on page seven—6, 18, 54, 162, 486—I multiply by three each time, since 6 · 3 = 18, 18 · 3 = 54, 54 · 3 = 162, 162 · 3 = 486. So the next item in the series would be 3 · 486 or 1458, a number that Rudolph wants me to find.

"Now, he wants me to subtract six from my new number, 1458. I wonder why. Maybe I should subtract six because the series starts at six. Anyway, 1458 – 6 = 1452, which he then wants me to divide by two, giving me 726, and that should be the sum of the four numbers. Yes, 6 + 18 + 54 + 162 + 486 is 726. It gives me the correct answer, but I wonder why. It looks almost like magic, but I'm sure that's not what it is, so there must be an explanation. Rudolph was mighty clever, but I doubt that he was any cleverer than I am.

"Maybe the trick is to divide by the number that is one less than the multiplier—the number that I used to get each of the next terms in the series. This time I multiplied by three, so maybe I divided by 3 – 1 = 2. That may be the explanation, but I don't have the time now to find out for sure. I think I hear Father coming home for dinner, and I can't let him find me working on this. I hope Rudolph will explain it on the next page. I wish I didn't have to stop now because this mathematics certainly is marvelous! I love it! Rudolph, I'll get back to you and your *Coss* as soon as I can."

A few days later, Jacob was working on a later section of the *Coss*. He found a problem on page ten that Rudolph says Pythagoras might have proposed 500 years before Christ. It was the story of a king who decided to establish 30 cities. For the first city, he would donate one dollar. For the second city, he would donate two dollars. For the third, he would donate four. For the fourth he would donate eight, and so on up to the thirtieth city, proceeding in this way with the powers of two. Today we would say that the first city gets 2^0 dollars, the second city gets 2^1, the third city gets 2^2—with each city getting the number of dollars represented by the power of two that is one less than the number of the city. In this way, the sixth city would get 2^{6-1} or 2^5 or 32 dollars, and so on, all the way up to the thirtieth city, which Rudolph says would require a total that we would write as 2^{29} and that Rudolph wrote as 536,870,912 dollars. However, since at this time the use of exponents was still several years into Jacob's future, he would have had no choice but to multiply by two repeatedly, just as Rudolph had done.

Jacob asked himself, "Is that really the total that I get when I multiply it out? No! It can't be that big! I guess I need to write it out all the way if I want to be sure." Then Jacob continued Rudolph's table. "For the eleventh city, I double the amount for the tenth city: $512 \cdot 2 = 1{,}024$ …. Now, continuing with my doubling, the fifteenth city gets 16,384, or twice as much as the fourteenth city,…." This was getting tedious, but Jacob was determined. "The twenty-ninth city gets 268,435,456, and the thirtieth gets … Yes, it gets 536,870,912. Remarkable! The amounts started so small, and see how quickly they became enormous!"

Jacob protested, "But these numbers are impossibly big! Pythagoras must have known that no king could have that much money to give to his towns. What a foolish king, and how wise Pythagoras was! Clearly Pythagoras and Rudolph want us to see how incredibly powerful a series of numbers like this can be. How can my father object to this?" Jacob asked himself. "I am supposed to be preparing for the life of an intellectual, and what could be more purely intellectual than mathematics?"

Jacob continued these studies diligently, and within six months he had mastered the *Coss*. What fun it was! And it was so much more exciting to him than pure philosophy! He was developing even greater respect for Plato—the purest of philosophers—who had recognized the purity and importance of mathematics so many centuries ago.

The professor in mathematics at the university knew very little more mathematics than Jacob did by now. His background was in philosophy, but since the position in mathematics was the only one that had been available when he had submitted his application, he had accepted it and had done the best he could. That was common practice at the university in Basel at the time—a professor took a chair in whatever field he could. All university professors had begun with a general philosophical background, many possessing only a veneer of specialization, and many hoped to change into a preferable—or perhaps better paid—field once a better position became available.

The truly great scholars in Europe in the sixteenth or seventeenth century did not make their careers in a university. Instead, they worked in the court of a king or a duke, who expected to derive some prestige for his enlightened court from them and who felt free to ask for an occasional invention or innovation from his resident scholars. By contrast, Jacob's professor at the university was not a great scholar. As was typical at the time, he struggled to handle a heavy teaching assignment, drawing on his limited background but hopeful that perhaps sometime in the future he would be able to pursue a truly intellectual career. As a professor, he was a workingman, condemned to long hours of teaching with only limited compensation.

Fortunately for Jacob, the mathematics professor at Basel was well enough informed to be aware of where Jacob could find some more advanced material in mathematics. He suggested that Jacob look into the writings of Pappus, who had lived in Alexandria on the Egyptian coast of the Mediterranean Sea in the third and fourth centuries A.D. Pappus' work was the most complete presentation of ancient Greek mathematics that was available in Europe at this time. Since the intellectuals of Europe before 1800 revered the Greeks as the greatest scholars ever, Greek mathematics was quite naturally the mathematics they would choose if they were to pursue mathematics.

"But where can I find Pappus' work?" Jacob asked. "Is it in the university library?"

"It should be," the professor replied. "I doubt that anyone has looked at it in many years—the dust is probably very thick on the volume—but the material inside is timeless. The dark ages of early medieval Europe are supposed to be behind us now, but I fear you will be joining a very small group of scholars who will actually be familiar with Pappus."

"Did you find Pappus difficult?" Jacob asked eagerly.

"Oh, I'm afraid I haven't read any of his work," the professor admitted. "I would be surprised if there is anyone in any of the Swiss cantons who has read Pappus."

"But you are the mathematics professor!" Jacob said. "Isn't this supposed to be the oldest and finest Swiss university? How can there be no one on the faculty who has studied mathematics?"

"Most scholars," the professor explained, "are far more interested in philosophy and theology than in mathematics."

"Those are the fields that my father wants me to concentrate on," Jacob admitted, "but I want to do more than that."

"Well, I'm afraid most people share your father's view today," the professor said.

"Do you suppose there might be someone in Geneva who has studied Pappus?" Jacob asked.

"It's possible," the professor said doubtfully. "Since my training was in philosophy, I have read very little mathematics, and I suspect the same is true of the mathematics instructor in Geneva. I can tell you for certain that no one on our faculty is well-grounded in mathematics."

"But why not?" Jacob retorted. "I can't think of anything that is more important."

"I would like to study it," the professor said, "but I simply don't have the time with all the basic courses that I have to teach. Perhaps after you complete your studies you could learn enough mathematics that you could offer the subject more completely than I do."

"I hope that I'll be able to do that," Jacob said.

"It would be wonderful if you could," the professor said.

"Do you suppose that Christoff Rudolph would have read Pappus before he wrote the *Coss*?" Jacob asked.

"I think that is unlikely," the professor replied. "I doubt that it would have been available to him when he was doing his work. I don't think that he could have found the works of Pappus anywhere north of the Alps, and I don't think he ever traveled to Italy. I believe Commandinus' Latin translation of this fourth century Greek work was published in Italy no more than a hundred years ago, and that would have been after Rudolph's time, and it probably wasn't available in any of the Swiss cantons or in Germany even then. I believe that our university library here in Basel bought a copy of Commandinus' Pappus sometime before I became a professor here. At least, it certainly should have bought it."

"Then isn't it strange that I was able to find the *Coss*?" Jacob asked.

"Not really," the professor said. "I suspect that is because the *Coss* is a basic textbook, which has some practical applications in the world of trade. Many businessmen are eager for their sons to be prepared for a life in commerce, and, as you have seen, the *Coss* has some material that businessmen can find useful. Pappus is different. He presents both geometry and logic—it is an interesting combination—with no obvious practical applications. Remember that Plato considered mathematics a part—and he meant an important part—of philosophy. However, I believe you will find it fascinating."

"Thank you," Jacob said, picking up his satchel and preparing to leave the interview.

"*Herr* Bernoulli," the professor added, "I just remembered that there was another mathematician, named Viète—a Frenchman who lived about a hundred years ago—who apparently did some interesting mathematics also. Unfortunately, I know nothing about him, and I have no idea where you could find his work. I have only heard his name. If you can find some of his work, I expect it would be interesting to you as well."

"Thank you. I guess I'll take a look at Pappus' work first if I can find it," Jacob said. "I'll have to wait a bit for Viète since my time is somewhat limited. Unfortunately, I am supposed to be concentrating only on philosophy. However, could you please tell me how to spell Viète's name?"

"He was a Frenchman. I think the French spelling of his name is V-i-è-t-e," the professor said, "but I believe I've also seen it spelled in Latin V-i-e-t-a. He would have written in Latin, of course, and that's the Latin spelling of his name."

"Thank you for the tip, Sir," Jacob said as he bowed politely to his professor and took his leave.

4
His Little Brother Johann "Helps" Jacob with Mathematics

In 1671, Jacob completed his master's degree in philosophy, having put off most of his further studies of mathematics until he had completed that crucial degree. He had satisfied his father by engaging in the expected debates, demonstrating beyond a doubt that he was an informed and articulate scholar of philosophy. The next step was to study theology in order to complete his licentiate in theology, the qualifying course of study for a Reformed minister. However, he had taken the time to find Commandinus' Latin translation of Pappus' work, the *Collection*, in the university library, and now he was ready to tackle it in what spare moments he could find.

Fortunately, his father was not at home this afternoon, so Jacob expected to be able to work in peace. He had the text open on the table in front of him as he was making drawings using a pencil and straight edge (a ruler), carefully following the steps in Pappus' argument. Although the text was accompanied by illustrations, Jacob found that the concepts were easier to follow if he actively constructed them step by step rather than simply looking at Pappus' ready-made drawings.

"Jacob," his four-year-old brother Johann scampered into the room and asked, "what are you doing? Tell me! Tell me! Please!"

"Jacob! Jacob!" Jacob's nine-year-old brother Nicolaus angrily shouted as he stormed into the room at the same moment. "Where did you get that paper? It's mine! Give it back to me."

"Go away!" Jacob said to both his brothers. "I'm trying to work!"

Nicolaus persisted: "Did you take the paper that I left out on the table? Father gave it to me, not to you!"

"I took only a few sheets," Jacob explained. "You still have lots of paper left. Go away and make your pictures. Are you planning to be an artist when you grow up? I can't believe that Father is encouraging you in that."

Nicolaus ran out of the room to see if Jacob had indeed left him enough paper. Jacob had to admit that Nicolaus was pretty good at drawing, although he was surprised in later years when Nicolaus actually became a respected artist.

Jacob then returned to his work, hoping for an uninterrupted hour or two for his studies.

"Jacob," little Johann persisted, "please tell me what you are doing."

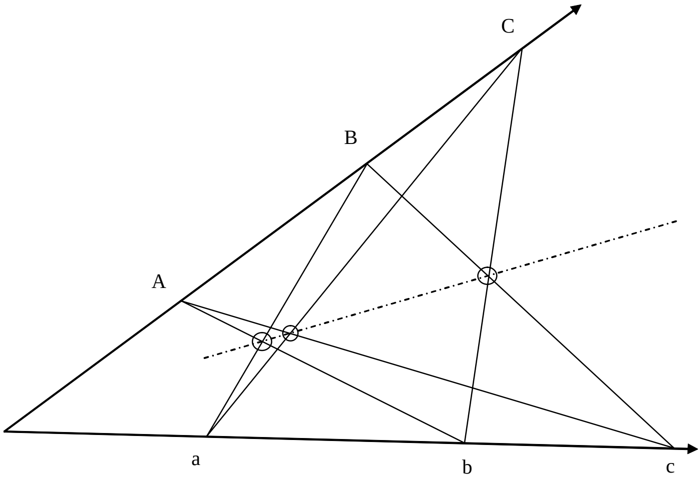
"You wouldn't understand," Jacob said. "It's mathematics, and it's a fascinating subject. Since you don't even know how to count yet, I won't bother to try to explain it to you. There is no way you would understand it. Go away, brat!"

"I do too know how to count!" Johann protested. "I can count all the way to 20: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20. See, I'm not such a baby!"

"You skipped 17—it should be between 16 and 18!" Jacob corrected him. "After you have finished learning to count and after you have learned basic arithmetic, I'll teach you some real mathematics, but you'll have to wait a long time for that."

"But Jacob," Johann persisted. "You're drawing something. I can make pictures too."

"No, Johann," Jacob said, "this isn't like Nicolaus' art. It's not just a pretty picture. This is a drawing of Pappus' Theorem. Look



Pappus’ Theorem: The three marked intersection points are all on the dotted line.

at what I’ve done here. See, I have made two straight lines that both start at the same point but go out in different directions from there. On this upper line, I have placed three points that I’m calling capital *A*, *B*, and *C*. On the lower line I have also placed three points, and I’m calling them lower case *a*, *b*, and *c*. You already know the alphabet, don’t you?”

“Of course I do!” Johann said. “It’s A, B, C, D, …”

“That’s enough!” Jacob snapped. “I’m trying to work.”

“Is it important to call the points by those letters?” Johann asked. “Couldn’t you use other letters—maybe *p*, *q*, and *r*—if you wanted to?”

“I suppose if I wanted to, I could. However, Pappus started at the beginning of the alphabet, so that’s where I plan to start too,” Jacob explained. “What I’m going to do now is to draw a line from capital *A* to lower case *b* and another line from lower case *a* to capital

B so that I can find the point where those two lines meet. This time I have to be careful to always work with only *a*'s and *b*'s, connecting each capital to the lower case of the other letter. This point that I've marked is the point that I want."

"Are you going to do the same thing with the other letters?" Johann asked.

"That's right," Jacob said. "I'll draw a line from capital *B* to lower case *c* and another line from lower case *b* to capital *C*, this time concentrating only on *b*'s and *c*'s, so that I can mark the point where those two lines meet."

"Why don't you draw a line from capital *A* to lower case *a*?" Johann asked. "You could do that, couldn't you? They're opposite each other too."

"I can't because that's not the way Pappus did it!" Jacob explained impatiently. "Pappus always deals with different forms of two letters at a time. That means that when he's dealing with *a*'s and *b*'s, he takes the capital of one and the lower case of the other, and then he reverses the process: he takes the capital of the other and the lower case of the first in order to locate his point."

"Okay, then what are you going to do next?" Johann asked.

"Now I'll draw a line from capital *A* to lower case *c* and another line from lower case *a* to capital *C* and mark the point where those two lines intersect," Jacob explained. "Careful! You just bumped my arm! Stand back."

"I'm sorry, Jacob," Johann said. "I didn't mean to. I'll try to be more careful. Now what are you going to do?"

"Now I'm going to admire my work," Jacob said. "Look at those three marked points. They are all on a straight line, and Pappus says it will always work out that way. Isn't that amazing?"

"Jacob," Johann asked, "when Nicolaus draws, he doesn't use a straightedge, and he sometimes uses pretty colors."

"I just told you! What I am doing is not art," Jacob explained. "This is mathematics. It is science. I am making a drawing so that I can see what the mathematics looks like. Nicolaus just makes pretty pictures. That is totally different. Mathematics is much more important."

"So what is mathematics?" Johann asked. "Your drawing doesn't have anything to do with counting, does it? I don't see any numbers at all."

"No. What I'm doing is part of geometry," Jacob said, "and geometry is a very important part of mathematics. I've got to work through more of Pappus' argument if I want to understand his proof."

"Do the letters have to be in the same order on both lines?" Johann asked.

"I think so," Jacob said. "I think it matters whether I put capital *A*, *B*, and *C* in one order on their line and then lower case *a*, *b*, and *c* in the same order on their line. Let's try changing the order and see what happens, just to be sure. This time I'll put capital *A*, *B*, and *C* in that order on the upper line, but lower case *a*, *c*, and *b* in that different order on the lower line. Quiet now! I need to do this carefully. Let's see if it works.

"Oh, no!" Jacob exclaimed. "The lines from lower case *b* to capital *C* and from capital *B* to lower case *c* don't cross when I change the order like that. If they don't cross, I won't have an intersection point to draw the line through. So I guess that shows that the order really does matter—I guess Pappus knew what he was doing. I wonder what happens if I make points capital *D* and lower case *d* on my original drawing and work with them the same way I did with capital *A*, *B*, *C*, and lower case *a*, *b*, and *c*."

"Why don't you try it?" Johann cheerfully asked. "Isn't that the best way to find out?"

"Okay, here it is," Jacob said as he continued to draw.

"They look as if they are on a straight line to me," Johann announced.

"Actually, it isn't perfect," Jacob admitted, "but maybe that's because my drawing is not as good as it should be. I think I'll try it again. I guess I have to be careful to always use the very center of each of those points." Jacob concentrated fiercely on his drawing as Johann impatiently waited.

"Yes! They are all on a straight line! You did it!" Johann shouted. "I knew you could do it. My brother, the math man!"

"What I am going to be is a mathematician," Jacob corrected him. "But wait! I'm not so sure that they're all on a straight line. But yes! Yes, I think they are too in a straight line! Look, Johann, if I hold this string over the points and then I pull it tight, all those points are under the string. That means they are all on a straight line."

"I like it, Jacob!" Johann said. "I think it's fun! Will you let me watch you do mathematics again?"

"As long as you don't bother me," Jacob said.

"I was good today, wasn't I, Jacob?" Johann asked.

"Yes, you were pretty good," Jacob said.

"Does Father know what you are doing?" Johann asked in a sweet little voice. "Does he know that you are going to be a mathematician?"

"Of course he doesn't, and you are not going to tell him," Jacob said. "If you breathe a word of this to anyone, I will never allow you to watch me do mathematics again."

"I won't tell," Johann promised. "You can trust me. I plan to grow up to be a mathematician too. Maybe I'll even be a better mathematician than you!"

"Highly unlikely!" Jacob snapped. "I have a head start on you, and all that will be left for you to do is to master what I choose to teach you. Now go away. I have more work to do, and I don't want your help this time. Go somewhere else and practice counting."

5
Having Completed His Studies in Philosophy and Theology, Jacob Moves On

Jacob's father arrived home one evening in 1676 after a long day at work. Only Jacob was around—none of the other children or his wife were at home—and this seemed like a good time for a serious talk with his oldest son.

"What are you planning to do now, Jacob?" his father asked. "You have completed your master's degree in philosophy and your licentiate in theology, you have two calls to become a pastor in the Reformed Church, you are 22 years old, and I think it is time for you to accept one of those calls in the Church. You have already distinguished yourself with two excellent sermons. I am very proud of you. There is no nobler calling than the Protestant ministry, and I believe you are ready for it."

"Oh, no!" Jacob said. "I can't do that—or at least not yet! Well, I suppose I could, but I'm not ready to take that step yet."

"So what do you plan to do?" his father demanded.

"I plan to continue my study of mathematics," Jacob explained. "I already know more mathematics than anyone else in Basel, so I must travel if I want to learn more. I need to find out if anyone in Geneva has studied mathematics. It is an incredibly exciting field, and I must learn more about it!"

"No! What kind of nonsense is that?" Nicolaus asked. "I have already told you that that is not my plan for you."

"But Father," Jacob Bernoulli protested, "I agree that theology and philosophy allow us to approach all of life more thoughtfully and nobly, and I have learned a great deal about them at the university as you wished. But if we stop and think a minute about our family history, you have to admit that working with numbers intelligently and accurately is what allowed our family to become successful importers of spices years ago. Without arithmetic, we would have failed then. What I have learned is that mathematics is far more than adding and multiplying. Just because you don't know anything about it does not mean that it is not important. You will see. I will travel and learn what mathematics has to offer now, and with my knowledge I will take it further than anyone today suspects is possible. I plan to be a great scholar."

"No!" his father said! "That is rubbish, young man! You are arrogant! … insufferable! It is true that our family has benefited from the arithmetic that has been passed down to us. You are right that it has allowed us to succeed in business, but there is no more to mathematics than that. I am your father, and you will do as I say."

"No, Father," Jacob said. "You must admit that our family has always survived by our wits—our wits strengthened by our knowledge and our integrity. Of course we need to have a firm moral foundation as well as knowledge of our culture, but if we are no more than moral people, we will lose out in the end. Remember, when your great-great-grandfather Jacob left Antwerp, he took a big chance. His father might not have approved of it, but clearly it was the right thing to do. You have to admit that his move to Frankfurt could have been disastrous. Our family's later move to Basel was chancy as well. Those earlier Bernoullis took enormous risks. Father, with all due respect, I would like to take a chance as well, and I believe the result will be similarly good."

"No, Jacob," his father Nicolaus responded, shaking his head sadly. "Certainly our ancestors' move from Antwerp and later from Frankfurt were wise moves, and I don't deny that arithmetic helped our family to establish a solid business. I suppose I have to admit that we have become one of the prominent trading families in Basel, at least in part because of our mastery of arithmetic. I never said that calculating is not important—of course it has helped us—but I am determined that you will have the life that I was not able to have.

"Your grandfather and his grandfather before him fought for our religious freedom. You will be the first in our family to pursue the life of the cloth, and you have completed the studying that you need in order to do that. It makes me proud to think of that. And a life in the Church will be well enough paid that you will be able to support yourself and a family comfortably. That is arithmetic that I can understand very well."

"But Father," Jacob said, "that is not what I want to do—at least not yet. Please allow me to travel to Geneva and then to France so that I can pursue mathematics. Just because you don't understand it does not mean that it is not important, and remember that we are talking about my life—not yours. Many of the men whom I have been studying with are going to travel for a couple of years before they settle down for their life work. While I am traveling, you may be sure that I will take advantage of opportunities to preach so that I will continue to build up a good reputation as a cleric as well. I promise you that I will make you proud before I am done."

"Well, I guess you may take a little more time before you settle down," his father said, "so long as it doesn't interfere with your real career in the Church."

"So you have decided to allow me to learn more about mathematics?" Jacob asked.

"You are trying to trick me into taking your side," Nicolaus barked.

"I must study mathematics. I must travel," Jacob Bernoulli informed his father. "I have the university degrees that you required me to get, but I am not willing to stop there."

"Poppycock!" Nicolaus Bernoulli fumed as he sat down at the table, pounding his fist as he continued to speak. "I can't see that your mathematics will have any application to your life in the ministry. And if you think that you would be able to support yourself and a family with a career in the university, you are wrong. Professors are the poorest of the poor. A parish priest has a far more comfortable life, earning more than twice as much as even the most famous university professor. You've seen them. They have a miserable existence. I have better plans for you."

"In fact, a pastor earns only half again as much, not twice as much," Jacob corrected his father, "but regardless, I must learn more mathematics. The mathematics that I want to study is more abstract than philosophy, and I believe it is far more important for the development of western civilization. Plato, the greatest philosopher of all time, would approve of my plans."

"Hrmmmmmpf!" his father grunted.

"I will depart for Geneva in the morning," Jacob continued. "I have made arrangements to tutor the children in the Waldkirch family there. One of the children, Elizabeth, is a girl who is blind. The father (a prominent businessman there) is convinced that all the children, including Elizabeth, are very bright. Since he wants me to teach Elizabeth to read and write and do arithmetic, in addition to teaching all the children such basic subjects as logic, physics, history, and all the rest, he needs a tutor who can be innovative enough to accomplish all that. He has learned that Girolamo Cardano (1501–1576)—a great mathematician in the last century—did some work on teaching a blind person to read and write.

"I have to admit that I had never thought before about whether it was possible for a blind person to learn to read and write, let alone how it might be accomplished. However, I have a description of Cardano's approach, and I'm hoping to improve on his methods. Although he was only partially successful in teaching his pupil how to read and write, I plan to do it right. I will succeed. I think this is an exciting project."

"I'm not impressed," his father muttered.

"Father, think about this a minute," Jacob said. "You want me to have a career in the Church, doing God's work on earth. You have to agree that teaching a blind girl to read and write is part of God's work also. Please give me some funds and the loan of a horse so that I may begin. After that I should be able to support my investigations in mathematics through tutoring. I must study with the great mathematicians of Europe. I will keep you posted on my whereabouts. Farewell, Father."

"Hrrmmmmmph," and Jacob's father left the room.

"Well," Jacob said to himself, "I guess the motto that I have chosen for myself fits: "*Invito patre sidera verso*—against my father's wishes I will study the stars." Jacob was comparing himself to Phaeton, the boy in Greek mythology who asked his father Helios, the sun god, to allow him to drive the chariot of the sun across the heavens for just one day. Although Phaeton's father had promised his son that he could have one wish, he never dreamed that his son would ask for this! It was a foolish wish, but the stubborn child reminded his father of his promise, and Helios felt impelled to keep his word. In the myth, since Phaeton was not strong enough to control the chariot of the sun—because unlike Helios he was not a god—the sun chariot was immediately in grave danger of crashing to the earth and destroying it. Zeus, the king of the gods, used his supernatural power and hurled a thunderbolt at Phaeton, killing him rather than allowing the rebellious boy to destroy the earth.

Like Phaeton, Jacob was sure that he could master his chosen chariot—astronomy and mathematics—but, unlike Phaeton, he would be able to reach for those stars in safety. There would be no need for Zeus or anyone else to interfere in his ambitious journey. Jacob was no fool, and his plan was something he knew he could carry out on his own. Jacob couldn't understand why his father refused to approve of the ideal life to which he was drawn—how could his father be so wrong? To Jacob, mathematics (and with it, astronomy) was the most beautiful subject imaginable, and he used his motto with relish for the rest of his life.

As he completed his studies, Jacob also chose a symbol to accompany his motto. It was the logarithmic (sometimes called equiangular)

Snail shell.

Jacob's seal, cloister of the Basel Münster.

spiral, which Jacob called the *spira mirabilis* [miraculous spiral]. As the size of the spiral grows (see picture), its shape remains the same. As the tangent follows the growing curve, the angle formed by the tangent and the curve's radial line remains constant. The chambered nautilus shell (or a snail's shell—see picture) is a famous example, formed by the shellfish as it grows larger and larger. Jacob wanted to have this spiral on his gravestone, although the actual spiral that appears there in the cloister of the Münster in Basel is only an approximation of it. Jacob's spiral is accompanied by the words in Latin, "*RESURGO EADEM MUTATA*" [Although changed, I shall arise again the same], as the curve does forever.

6
Jacob Travels to Geneva and Meets Elizabeth Waldkirch and Her Family

The three-day trip from Basel to Geneva took Jacob first through the Swiss towns of Biel and Neuchâtel, where he spent the night in a small inn, making arrangements for his horse to be well fed and well rested before the next long day on the road. The second day he traveled along the beautiful lake Neuchâtel and then on to the city of Lausanne. He was impressed with the vast lakes he found and with sailing boats skimming across the surface. From his childhood, he had known Basel's Rhine River with its powerful current. Although he had often crossed Basel's mighty Rhine in the small ferries that were powered only by the force of the river's current, and he had seen the large river boats that carried goods up and down the great river, these placid lakes were new to him. When he and his horse stopped along the shore of a lake to rest, Jacob dismounted and just gazed across the wide expanse of still water. Once, he even found people playing in the water, some of them apparently floating on its surface. Was that what people called swimming? Although he was a strong young man, he would never attempt to fight the powerful current of the Rhine River in his home city. He knew that he was no match for it! Perhaps it was different in a lake—the people that he saw swimming did not look as if they were any stronger than he was.

From Neuchâtel on, he found people who spoke only French, so it was a good thing he had spent some time working to improve

Rhine River at Basel.

his French before he set off on this trip. At Lausanne, Jacob caught his first glimpse of the snowy Alps. The dazzling Mont Blanc looked as if it were made of the purest salt! Here, he and his horse spent the night in another small inn before an early start on his final day of travel along Lake Geneva, bringing him by mid afternoon to the city of Geneva, the refuge of John Calvin, the founder of a dominant Evangelical Church in Switzerland. Everywhere he looked, there were spectacular mountains such as he had never even imagined. He rode his horse over the bridge that spanned the Rhone River at Geneva, and finally reached his destination.

"You must be *Monsieur* [Mister] Bernoulli," *Monsieur* Waldkirch greeted Jacob in French. "I am delighted that you were willing to come to Geneva to work with my children."

"Thank you so much for inviting me here!" Jacob exclaimed, also in French. "I have to admit that I have seen sights that I never dreamed of on this trip. I had no idea Geneva was such a beautiful city!"

"Yes," *Monsieur* Waldkirch said, "the Rhone River is nothing like your powerful Rhine, but our river and our lake have their charms. Did you know that after Lake Balaton in Hungary, our Lake Geneva is the largest lake in all of Europe? However, you didn't come for a lecture on the beauties of Geneva! Please allow me to begin by welcoming you to our home."

"Thank you so much, *Monsieur* Waldkirch," Jacob said. "I expect I will be learning much about your city during the time that I will be here. I should tell you that I am truly delighted to accept the challenge of teaching your children. I am particularly intrigued with the prospect of teaching Elizabeth. I expect we will all get along splendidly."

"Well, I hope you will have great success," *Monsieur* Waldkirch said. "I think you will find that Elizabeth is extremely bright. Are there any supplies that I need to arrange for you?"

"Yes, *Monsieur*, I'm afraid there are," Jacob said. "I will need to find a carpenter or wood carver who can make me models of the letters and numbers so that your blind daughter and I can begin to work. At first her learning will have to be exclusively tactile—by feel. Do I understand that she speaks German as well as French already?"

"Oh, yes," *Monsieur* Waldkirch said. "I'm pleased to say that she seems to have a real flair for languages. But, let me ask you if you would prefer to teach the children in German rather than in French."

"Oh, no," Jacob quickly replied. "I believe it will be best for them, particularly for Elizabeth, to learn at first in their native language, and I think my French is up to the task."

"Yes, *Monsieur*," *Monsieur* Waldkirch said, "your French is excellent. I agree that it would be preferable if you can teach her in French if you don't mind. I should tell you that Elizabeth has a superb memory, and that has always been a real advantage for her. We never need to tell her anything more than once.

"However, I should have thought to arrange for the wooden alphabet and numbers before you arrived. I'm so sorry! But I guess it's too late now. Now that you are here, perhaps you would like to make the arrangements yourself since you have a better idea of what you need than I would. My friend Simon Cartier is a carpenter and wood carver who lives on the road into the city, and I think you will find that he does excellent work. You must have passed his shop on your way today. Just tell him what you need and ask him to put the charges on my bill."

"Excellent," Jacob said. "The letters and numbers will need to be nicely finished, of course, so that your daughter can comfortably trace the shapes with her fingers. I don't want her to get a splinter in her finger! Shall I go talk with *Monsieur* Cartier this afternoon? I really cannot begin with *Mademoiselle* until I have those models."

"Of course," *Monsieur* Waldkirch said, "if you are sure you are not too tired from your journey. I'll ask the groom in the stables to provide you with a fresh horse (yours must be exhausted after three days of travel) and directions to find *Monsieur* Cartier's shop.

"*Monsieur* Cartier," Jacob began as he entered the wood-working shop, "my name is Bernoulli, and I will be tutoring *Monsieur* Waldkirch's daughter Elizabeth. He thought you would be able to make the supplies that I need."

"I'm so glad you are going to work with little Elizabeth! What a charming child!" Simon Cartier said. "I think you will find that she is a very clever pupil. I'm sure her father has told you that she is very bright, and that is no exaggeration. What would you like me to make for you?"

"What I need is a set of letters and numbers made of wood so that she can feel the shapes and can get to know the symbols," Jacob explained. "If possible, I'd like you to make two of each letter and number, each on its own rectangular block of wood, all the same size and about this thick [Jacob showed a length of about a half inch between his thumb and finger], with the letter or digit carved out on one face of the block so she can feel the shape. I would imagine it will be easier for you if you don't make them too small. However, if it is at all possible, I would like them to be small enough to fit into a cloth bag. The blocks will also need to be sanded very smoothly so that they are a pleasure to touch."

"This sounds very sensible to me," *Monsieur* Cartier said. "As a woodworker, I love the feel of a beautifully sanded piece of wood! You should realize that this will be a labor of love for me—I am very fond of Elizabeth."

"I'm so glad!" Jacob said. "In addition to the digits and letters, I will also need some open boxes, a size that will allow one digit or one letter to fit perfectly into each box. That way I will be able to teach *Mademoiselle* how to form larger numbers and words so that she can get the spacing right," Jacob said. "For arithmetic, I'll need a box for the ones' place, a box for the tens' place, a box for the hundreds' place, and so on. For words, I guess I'll need even more boxes, but I expect we'll be able to use the same boxes for both numbers and letters. Do you think you can make all of those?"

"I'm sure I can," *Monsieur* Cartier said. "How many boxes do you need, and how soon do you need all these things?"

"I think 30 boxes should be enough, because once she understands the spacing she should be able to move beyond the boxes," Jacob said. "I'm afraid I would like to have everything as soon as possible because I really cannot begin my work with her until I have them. Perhaps you could prepare one set of the numbers and a few of the boxes first, so that we can get started on arithmetic. Then you could complete the rest of the sets while I'm working with her on the numbers. I imagine it will take her awhile to learn them."

"Would Monday be soon enough for the numbers and the first boxes?" Simon asked.

"Yes, Monday will be fine," Jacob said.

"By the way, do you need both capital letters and lowercase letters?" *Monsieur* Cartier asked.

"Yes, I will need both," Jacob said, "but I think one set of capital letters will be enough. However, I think there will not be such a rush on the letters. I suspect arithmetic will be a real challenge for her."

"*Monsieur* Bernoulli, I think you will be surprised at how quickly she learns," *Monsieur* Cartier said. "She is an unusually intelligent girl. I'll have my man deliver one set of numbers and several boxes to you at the Waldkirchs' home on Monday morning, and I'll try to have the letters and the rest of the numbers as well as the rest of the boxes ready by the end of the week. I like this project very much. I assume I should put this on *Monsieur* Waldkirch's bill."

"That's what *Monsieur* Waldkirch asked us to do," Jacob said.

"Good," *Monsieur* Cartier said. "Shall I ask my wife to make a bag for the pieces?"

"That would be wonderful, *Monsieur* Cartier! Thank you!" Jacob said as he remounted the borrowed horse and set off once again for the Waldkirchs' home.

When Jacob joined the family for supper that evening, he met all the children as well as their mother, *Madame* [Mrs.] Waldkirch, for the first time. The atmosphere in the home was warm, and Jacob was impressed with how poised Elizabeth was. He learned that she had lost the sight in both eyes because of an infection just two weeks after she was born. This meant that she could never remember seeing anything. However, she handled the dishes on the table easily, never spilling anything. All the children were articulate, carrying on a conversation in both French and German with no trouble. In fact, Jacob had to admit that Elizabeth's German was at least as good as his French. When he commented on this, her father explained that she could also speak Latin. Jacob decided that his assignment with this very bright child was decidedly possible, and he found that he liked the Waldkirch family very much. The family seemed happy, with lots of good fun as well as serious talk during the meal.

7
Jacob Teaches Elizabeth Waldkirch to Read and Write Numbers and Words

When the digits and boxes arrived on Monday morning, Jacob was delighted with them. All the surfaces were beautifully smooth, all the edges and corners had been expertly rounded off, and the drawstring bag was beautifully finished as well. Jacob began to work with Elizabeth at once. He gave her the digits one at a time, encouraging her to handle them for long enough to learn their shapes well. Fortunately, she already knew how to count and do simple arithmetic in her head.

He urged her to be patient at first, but he soon realized that Elizabeth had learned about patience from an early age. Jacob was the one who needed to be reminded about patience. This was his first experience as a teacher other than his informal sessions with his brother Johann. He was determined to succeed, but he needed to remind himself repeatedly that what was obvious to him wasn't necessarily obvious to her.

"*Mademoiselle*," Jacob said, "First, you will need to learn to recognize the shapes of all the digits. Please note that the digit 1 is a straight line with just a little hook at the top. Can you feel that?"

"Yes, *Monsieur*," Elizabeth said.

"Now I want you to feel the digit 2. It has a straight line across the bottom, but then it curves from the left end of the base up to the right and then around to the left, making a graceful loop. Do you feel that? Wait a minute! You do know your left from your right, don't you?"

"Yes, *Monsieur* Bernoulli," Elizabeth said. "This is my right hand. But excuse me for asking, *Monsieur.* What do you mean by digits? Is digit just another word for number?"

"No, *Mademoiselle,*" Jacob said. "There is an important difference. The digits are the symbols that we use to write the numbers. We have ten digits—0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. They are the symbols that you are learning now. I'll teach you how we use the digits to construct numbers as soon as you know the digits."

"But my father never used the word digit with me," Elizabeth protested.

"That's because he was not teaching you to read or write," Jacob explained. "If you are going to read and write numbers, you must begin with the digits. Until you learned to read and write numbers, however, there was no need to distinguish between digits and numbers."

"Thank you, *Monsieur,*" Elizabeth said. "I want to learn all of this well and quickly."

"Good for you!" Jacob said.

"Thank you, *Monsieur,*" Elizabeth said, "but can you tell me this: Is the digit nine just the upside down version of six? Can I make a nine by simply turning the six upside down?"

"Yes, I suppose you could," Jacob said, surprised at that obvious fact that he had never considered seriously before. "Can you see how both 6 and 9 curl around into themselves? They really form a spiral, a shape that I find very appealing."

"Yes, *Monsieur,*" Elizabeth said, "I like that too, but now could you please tell me about zero? It seems to be round with a hole in the middle. My father told me about zero, but I have never been able to understand why we need it."

"Actually, it's not completely round. It is really somewhat longer from top to bottom than it is from left to right. Please take this zero in your hand. Can you feel that difference? " Jacob asked. "It's what we call an oval. However, I should also answer your excellent question about the meaning of zero. The number zero simply means that we have none of the thing at all. If you have zero dolls, that means you don't have any, but you already knew that!"

"Yes, *Monsieur*, I know what it means to have no dolls or no bread, but why do we need a symbol for it? Why do we bother to count something if it isn't there?" Elizabeth asked.

"Sometimes we need to explain that a container or a group is empty, and the number zero is useful for that," Jacob explained, "but the digit zero is really far more useful than the number zero. However, I'm getting ahead of myself. I'll get to that a bit later."

"Of course, *Monsieur*. So you are telling me that zero does more than tell us that we don't have something. Thank you, *Monsieur*," Elizabeth said. "That's something I had been wondering about. I can wait for you to tell me more about it later, but please don't forget."

"Don't worry about that, *Mademoiselle*!" Jacob said. "I consider the digit zero extremely important. Now, let's review the digits one more time. What is this digit?" and he handed her a block.

"That is five, isn't it?" Elizabeth asked.

"That is correct, *Mademoiselle*," Jacob said. "Now, how about this digit? No, *Mademoiselle*, you must hold it right side up. It does make a difference."

"Yes, *Monsieur*. I'll try to be more careful," Elizabeth said. Then she reached over to where Jacob had placed the rest of the digits on the table—she knew exactly where they were—and she named each digit correctly as she picked it up, this time being careful to hold each digit right side up.

"*Mademoiselle*, I believe you know all ten digits now," Jacob said.

"I think so," Elizabeth said, "and I like them."

"Good. So now we can move on to the construction of numbers," Jacob continued, "and this is where the distinction between digits and numbers is important. You see, our number system uses place value—the location of a digit in combination with the value of the place in the number—which go together to tell us what a number is worth. The word for digit in French [*chiffre*] comes from the Arabic language. We have the Arabs to thank for our number system, so it is appropriate for us to use the Arabic word for the symbols. You might be interested that the German word for digit also comes from the same Arabic root."

"So, were the Arabs the first people to write numbers, *Monsieur*?" Elizabeth asked.

"No, the ancient Sumerians and ancient Egyptians wrote what I would have to describe as primitive number symbols using a combination of dots and lines many centuries before the Arabs," Jacob said. "The ancient Greeks and Romans wrote numbers also, but they used letters from their alphabets, and they also did not use place value in the modern sense. Greek and Roman written numbers were very awkward and often involve many symbols. Arabic numbers are vastly superior. We are very fortunate to have our number system. It makes calculating easy."

"That is very interesting," Elizabeth said. "Perhaps sometime you could tell me how the Greeks and Romans wrote numbers."

"I could do that, but I think we need to work with our own number system first. We work from the right as we construct a number," Jacob said. "The place on the right is the ones' place. We will indicate that by this first box. Please touch it with your hand, *Mademoiselle*. The next place, just to the left of the ones' place, is the tens' place. A digit in the tens' place is worth the value of the digit multiplied times ten in exactly the same way that a digit in the ones' place is worth its name times one. Does that make sense to you?"

"Yes, *Monsieur*," Elizabeth said. "Does that mean that the third box is the hundreds' box?"

"That's right," Jacob said. "What do you think a three would be worth in the hundreds' box?"

"It would have to be 300, wouldn't it?" Elizabeth asked.

"That's right!" he said. "Now the amazing fact about our number system is that we can write any number, no matter how big or small it is, using only these ten digits and however many boxes we need."

"Is this where we come to the use of the digit zero?" Elizabeth asked, unable to control her curiosity any longer.

"That's right, *Mademoiselle*," Jacob said. "What we need the digit zero for is to indicate that a box is empty. Since most people write numbers without boxes, we need a symbol to show that a given place is empty. So if there is a zero in the ones' place and a five in the tens' place, that means we have five tens and zero ones, so that number would be the number 50. Does that make sense?"

"Yes it does, *Monsieur*," Elizabeth said. "Would we write the number 500 by putting a five in the hundreds' place and then zeroes in the tens' and ones' places?"

"That's right, *Mademoiselle*," Jacob said. "Shall we try another number now?"

"Yes, please!" Elizabeth said.

"Okay, I have put a digit in each of these boxes," Jacob said. "Remember that the box on the right is the ones' box, the box in the middle is the tens' box, and the box on the left is the hundreds' box."

"Where should I start?" Elizabeth asked. "Should I start on the right?"

"Yes, let's do that for now, although after you have identified the parts of the number, we will actually read the entire number from the left," Jacob said.

He was delighted as he saw how quickly she figured the numbers out, correctly reading four- and five-digit numbers within only a few minutes. Before the end of the week, she was doing serious arithmetic with her boxes of numbers and even writing them on paper with a piece of charcoal. Jacob was pleased to see that her numbers were perfectly clear to any seeing reader. Filled with excitement, Elizabeth quickly took the paper and ran to show her mother what she had done. It was an amazing accomplishment! Her mother was quick to tell her that she knew her father would also be thrilled. After *Monsieur* Cartier's man delivered the letters, reading and writing proceeded just as quickly, and within several months both Jacob and Elizabeth were delighted. She truly could read and write.

One morning after they had worked for several hours together, Jacob said to Elizabeth: "*Mademoiselle*, I have a question for you that has nothing to do with reading and writing, but it is something I have been wondering about. I'm afraid it is rather personal. I hope you don't mind."

"Of course not," Elizabeth said. "You have answered all my questions, so there is no reason for me not to answer yours. What do you want me to tell you?"

"Thank you. I'm curious about how you dream when you are asleep," Jacob said. "When I dream, I see things in my mind. Can you tell me what your dreams are like?"

"Oh!" Elizabeth said. "I never thought about that. I guess I don't see things in my dreams the way you probably do, but things do happen. In fact, I think my dreams have been changing since you have been teaching me. I sometimes find myself handling the shapes of the letters and digits in my mind as I dream. It is almost as if I was awake and using them!"

"That is very exciting, Elizabeth!" Jacob said. "That means that in fact you are seeing. Thank you so much for telling me about that."

Because the children could not be expected to spend all their time on their lessons, Jacob had the occasional afternoon to himself. Some days, after a long morning of tutoring, he took his horse into town so that he could introduce himself to scholars at the academy in town and find out what mathematics resources were available. He was pleased to get to know several instructors and students in mathematics. However, although some of them seemed to be interested in mathematics, Jacob soon realized that he was far ahead of them all.

Explaining the *Coss* and Pappus to his new colleagues was challenging, but Jacob realized that it was helping him, too. In the process of explaining the subject, he was coming to understand it at a deeper level. To please his father, Jacob also engaged in several debates with theologians in the city and even delivered a sermon in one of the churches in town.

During his 20 months in Geneva, Jacob also had the chance to study Cardano's *Ars Magna* [*Great Art*, or the rules of algebra], which he was able to borrow from a friend of his employer in Geneva. It was this gentleman who had told *Monsieur* Waldkirch about Cardano, the mathematician who had first attempted to teach a blind person to read and write. Although Cardano's mathematics book had been published in Basel in 1570, Jacob had not been able to find it in the Basel University library.

During his time in Geneva, Jacob also developed some skill at the game of tennis. While still at home in Basel he had occasionally picked up a racket, but it was only in Geneva that he was able to play tennis regularly and develop his physical coordination for this sport. Since the local sport club had fine tennis courts, Jacob had many occasions to play. In later years, he explored probability as it relates to games such as tennis.

8
Sundials, and Tutoring in France

At the end of his time in Geneva in the spring of 1677, Jacob, who was then 23 years old, received word of a position in France tutoring the children of the Marquis de Lostanges. The marquis offered to pay for his journey by carriage to Nède near Limousin in south-central France, where once again he would serve as a tutor. Since Jacob had determined that he needed to journey to France to continue his study of mathematics, he was pleased to take advantage of this offer. Once again he found time to explore mathematics during some free afternoons. At this time, Jacob made a serious study of sundials and their construction, perfecting a method for determining the angle for the gnomon (the rod that creates the shadow from which the time is read) to accommodate the latitude of a specific location based on the tilt of the earth at that point. The angle is critical if the sundial is to be usable during daylight hours throughout the year. A properly constructed sundial's only limitation is the need for clear sunny weather and a southern exposure.

"*Monsieur* Bernoulli," the marquis asked him one afternoon, "Could you tell me about that table you have there?"

"Yes, *Monsieur*," Jacob replied. "This is a table showing the angle at which the gnomon of a sundial must be mounted at any given latitude in order to construct a reliable sundial. It is based on the inclination of the earth with respect to the sun. I have read about sundial construction, and I was curious to see if I could generalize the technique. It shouldn't be necessary to start from scratch with the calculations any time we want to place a sundial in a garden."

"Fascinating!" the marquis said. "And where did you find the table, or did you make the table yourself?"

"Yes, *Monsieur*," Jacob said, "as far as I know there was no such table available, so I made it myself. I have been doing the calculations for the table during my free moments over the past few weeks, using astronomical data to find the exact angle required at each of the latitudes. Apparently, no one else has taken the time to do that. Depending on the latitude of a location, I can easily get the angle just right, so that using my table I can construct a reliable sundial to be placed anywhere in France or the Helvetian Confederation."

"So, what would be the correct angle for a sundial here in Nède?"

"Here it is," Jacob said as he quickly scanned the table and sketched the angle.

"Would it be possible for you make a sundial that could be placed in my garden here in Nède?" his employer asked. "I would be delighted to pay you for it."

"Well, it would have to be in a location that gets sun throughout the day, possibly a wall with a southern exposure or an open spot in the middle of the garden away from large trees and buildings that might cast inconvenient shadows," Jacob said. "I must say, it sounds like an interesting challenge. I'd be glad to do it. I would love to put my table to a practical test."

"Then come over this way, please," his employer said. "I think I have a wall in my garden that would be just right." Suddenly the marquis noticed that Jacob was walking very slowly—he was not able to keep up with him. "Are you coming, *Monsieur* Bernoulli?"

"Yes, *Monsieur*, I am coming," Jacob said, "but I'm afraid I can't walk too quickly. Ouch! My toe!"

"I'm so sorry, *Monsieur*," the marquis said. "I can tell that you are in terrible pain. Perhaps we should talk about the sundial another time."

"Oh no, I'm fine," Jacob said, hobbling as quickly as he could and trying not to show his distress. At this time Jacob was beginning to suffer from serious health problems, often resulting in difficulty walking. Lately he was finding the game of tennis entirely too painful. He had been soaking his foot morning and evening all week, but still his big toe was badly inflamed. For the rest of his life, he suffered from severe pain in his legs and feet, probably the result of gout and perhaps from scurvy (a deficiency of vitamin C) as well. At that time, the standard European diet during the winter included very few fruits and vegetables, the natural sources of that vitamin. Unfortunately, Jacob's pain sometimes distracted him from the scientific research he wanted to pursue, and eventually his illnesses would cut short his life.

"*Monsieur*, is this the wall you were talking about?" Jacob asked stoically.

"Yes," the marquis said. "What do you think?"

"I think it would be perfect," Jacob said, holding his arm at the approximate angle so that he could see its shadow.

"That will be wonderful!" the marquis said enthusiastically. "I have wanted a sundial here in my garden for several years, but until today I didn't see how I could get one. You are a very clever young man!"

"I'll need to have a blacksmith make a straight rod for me," Jacob continued, "but that should not be difficult for him. Do you have a mason who could implant the rod at the angle that I specify so that I can proceed from there and place the markings on the wall?"

"Of course! How long should the rod be?" the Marquis asked.

"I would think about as long as the distance from a man's elbow to the tip of his finger," Jacob said. "I wouldn't be surprised if the smith even has such a rod on hand."

"I'll instruct the smith to have one ready for me by early tomorrow morning," he said. "My mason is coming tomorrow afternoon. I'll send word to them both to make those preparations."

"Then the rest should be no problem," Jacob said. "I should be able to make my plans this evening."

"Good. I'll send a messenger to the blacksmith now," the marquis said, "and then I'll send a message to the mason to pick up the rod on his way to my house tomorrow."

"That should be fine," Jacob said. "I will also need some paint and a fine paintbrush so that I can do the markings on the wall. Shall I write something on the dial, perhaps *tempus fugit* [Latin for 'time flies']?"

"That would be good!" the marquis said enthusiastically, "or how about '*sic vita fugit*' [thus life flies] for a little irony? You are too young for such thoughts, but by the time you are my age, you will begin to think about the passage of the years as well as the hours!"

Jacob's sundial on the wall of the marquis' house was a charming work of science and art, which allowed the family to know the precise time on any sunny day. Visitors to the garden were always impressed (the marquis was delighted to demonstrate that he possessed the latest technology), and one neighbor was so impressed that he asked Jacob to make him a sundial as well. Chuckling at the message on the marquis' sundial, he suggested that his sundial's message could be the simple "*Je ne compte que les heures ensoleillées*" [I count only the sunny hours]. Jacob was delighted.

The neighbor chose an open spot in his garden for which Jacob designed a handsome sundial to be constructed on top of a large flat stone. Once again, Jacob needed the help of a stone mason and a blacksmith to erect the flat stone in the garden and to plant the gnomon in the stone at the precise angle that Jacob gave him. Once it, was mounted, Jacob made the markings on the stone to designate the hours and then wrote the saying. Jacob was developing a reputation in the neighborhood not only as a savant, but also as a clever but practical scientist.

Once again keeping his word to his father, Jacob preached several sermons in area churches around Nède. By this time, Jacob's French was excellent and his sermons were well received. Although

Jacob had already decided that a life in the ministry was not his first choice, he was enough of a realist to keep his options open. He had made a serious promise to his father which, as an honorable young man, he intended to keep. This was wise since he had not yet met anyone who had actually made a career in mathematics.

At this time Jacob began his *Meditationes*, his scientific diary, in which he recorded his explorations into mathematics and physics. He wrote it in Latin—technically a "dead language" for more than 1,000 years, but still the living language of scientists throughout Europe. As a well-educated young man, Jacob wrote easily in Latin, often even writing poetry in Latin for his own and his friends' amusement.

In 1679, Jacob journeyed farther west to the city of Bordeaux, to tutor the son of a local official there—a notary who was certified to prepare and sign off on official documents. At this time Jacob was accumulating as much money as he could from tutoring and consulting about sundials, meeting with scientists wherever he could find them, and reading anything he could find on mathematics.

Jacob was fascinated to observe firsthand the phenomenon of tides as he walked around the seaport of Bordeaux. He had learned about high tide and low tide as a school boy in his studies of geography, but as he watched the water level actually rising and falling in the harbor, Jacob was astonished. The Rhine River in Basel certainly did not behave this way! The tides, which at first had struck him as unpredictable, were something the men who worked in the harbor dealt with every day. Their understanding, however, was only at a practical level. Jacob wanted to understand why the tides happened, and he wondered how precisely they could be predicted. Jacob was an ambitious young man.

After completing his duties in Bordeaux, Jacob had accumulated enough money that he could then travel to Paris and study full time. His first goal was to read the works of Descartes, who Jacob knew was the most esteemed mathematician of the current century—probably the most important French mathematician ever. He had been trying everywhere to find someone who had a copy of Descartes' work *La Géométrie*, which had been published in French in 1637 and later in Latin in 1649 and 1659—translated by the Dutch mathematician Frans van Schooten (1615–1660). Jacob still had not found a copy of this work in either French or Latin.

9
Jacob Meets with Mathematicians in Paris

In 1680 at the age of 26, Jacob traveled to Paris, where he arranged to meet with several philosophers and mathematicians. He was particularly impressed with Nicolas Malebranche (1638–1715), a nobleman who was a priest and a philosopher and whose library included the works of François Viète (1540–1603) and René Descartes (1596–1650). Malebranche had spent much of his life studying Descartes, concentrating on both his philosophic works and his mathematics. Malebranche was the kind of scholar Jacob had been hoping to meet.

"*Monsieur*," Jacob said, "I have heard of Descartes, whose work I have been eager to read for some time, but all that I know of Viète is just his name. Did you say that he came before Descartes? Do you think that it is important that I start with the mathematical writings of Viète or could I simply begin with Descartes?"

"Well, *Monsieur*," Malebranche replied, "there is no denying that Viète was important. He did his major work about 50 years before Descartes, and I think it is clear that Descartes learned from Viète's work, although I don't know that he ever admitted any debt to him. There is one good story that you should hear about Viète if you know nothing about him.

"In 1590, during a war with Spain, Henri IV, the king of France, obtained some intercepted letters from Spain written to nobles in the French court. These letters were written in cipher, and the king was determined to find out what they said, since he thought they might have information that would be valuable to him. If they didn't, the king asked, then why would anyone have bothered to write them in cipher? The cipher was difficult, and no one among Henri's advisors could even begin to decipher them.

"However, in his court Henri also had a mathematician (that would be Viète), so he asked Viète to try to decipher the letters. After a few months' effort—it was certainly a difficult code!—Viète cracked the cipher, and Henri was able to foil the Spanish king. The Spanish king, for his part, couldn't believe that anyone could decipher his message, and he immediately accused Henri of using witchcraft! The members of the Spanish court had sworn that no one, anywhere—particularly not a bungling Frenchman (!)—would ever be able to decipher it. I love that kind of witchcraft, don't you?" Malebranche asked Jacob.

"That is a wonderful story!" Jacob said. "I wonder if it would impress my father, who has fought against my studies of mathematics since I began studying at the university."

"I'm sorry he's done that," Malebranche said. "I hope you won't ever let him keep you from it. I cannot imagine life without mathematics."

"Well, I'm here, and I'm learning mathematics!" Jacob said. "You may be sure that I will study mathematics regardless of what my father says."

"That's good," Malebranche said. "Getting back to your question, I suppose it is reasonable to say that there is no need for you to read Viète's work yourself, although it would be unfair not to give him credit for inspiring some of Descartes' important work, including the use of letters to represent known and unknown quantities. Viète's plan was to denote unknown quantities by vowels (*A, E, I, O*, and *U*) and known quantities by consonants (*B, C, D, F*, etc.), while Descartes chose instead to use the letters at the end of the alphabet to stand for unknown quantities (*x, y*, and *z*) and letters at the beginning of the alphabet to stand for known quantities (*a, b*, and *c*). I, personally, doubt that the choice of letters is significant, although Viète was limited to five unknowns since there are only five vowels, and I suppose that could possibly pose a problem sometime. We truly have Viète and Descartes to thank for what I like to call literal algebra—algebra using letters—which I believe will soon be all anyone will ever use for algebra. The *cossists* are already folding up their tents—they know they have already lost. You will see. Literal algebra is truly the mathematics of the future."

"I can hardly wait to get started!" Jacob said. "I have to admit that I have always found Rudolph's words and abbreviations cumbersome."

"You are right about that," Malebranche continued. "Another difference between Descartes' and Viète's algebras is that Descartes used a superscript—an exponent—when he wanted to indicate $x \cdot x$, writing it as x^2 or $y \cdot y \cdot y$ as y^3, and I believe that may be significant."

"So the exponent tells how many times the quantity is multiplied times itself? I like that!" Jacob said. "Rudolph could have used that in his *Coss*!"

"Yes," Malebranche said, "and Viète only 50 years earlier still used only verbal or syncopated symbols such as "*A quadratum*" [*quadratum* is Latin for squared] or "*A* quad" (in much the same way that Rudolph did in the *Coss*), whereas Descartes wrote as it as a^2. I suspect Descartes' notation will be the one that survives, but we'll have to wait and see. Descartes' work is certainly much better known than Viète's today, and probably with good reason. I believe we are working at a very exciting time in the development of mathematics. Do you suppose someone 200 or 300 years from now will simply consider Descartes' work the norm?"

"That is possible," Jacob said. "But I have to admit that I have struggled simply to find a copy of Descartes' work for most of a year, so it still isn't as easy to find as it should be."

"No, it's a pity," Malebranche agreed.

Jacob continued, "I think the exponent—is that the word you used?—sounds like an excellent idea as a substitute for *quad*, but it may take me awhile to get used to it. I would think it would be faster, and certainly it's a pity to slow down our mathematics just because of inconvenient notation. *Coss*' abbreviations were an improvement over the verbal mathematics of the classical mathematicians."

"Would you like to hear an interesting little tidbit about Descartes' use of letters for variables?" Malebranche asked.

"I would love to!" Jacob said. "You are a gold mine of information on mathematics!"

Malebranche continued, "I have read that Descartes planned to use the letters *x, y*, and *z* to stand for his unknown quantities, and he hoped mathematicians would use a variety of those letters. However, he lost on that point. Apparently his printer had some difficulty with the availability of letters. He found that he was running low on his supplies of *y* and *z*. As you know, the French language uses those two letters a great deal, but it uses the letter *x* much less often. So the result is that the printed version of Descartes *Géométrie* uses *x* as a variable most of the time. It was a practical solution to a practical problem, having no mathematical significance at all. The Latin translation of Descartes' work has continued that, even though the printer's problem does not arise in Latin. I was amazed when I read this. I wonder if mathematicians will continue to use mainly *x* for the variable in the future. That would not have pleased Descartes, may he rest in peace!"

"That is very interesting!" Jacob said. "I wouldn't have expected it to be a practical issue, but I can see the printer's problem. Now in German, we don't use the letter *y* anywhere near as much as French does, and we probably use *x* even less than the French, so a German publisher, given a choice, might have been willing to alternate the letters *x* and *y*, satisfying Descartes at least in part. Interesting! But wait! Why didn't Descartes write in Latin? That is the language of science."

"I don't know why, but he didn't," Malebranche said. "He wrote in French, although now his work has been translated into Latin so that mathematicians throughout Europe can read it where it is available. You have unfortunately found that availability is a serious problem. I'm sure Descartes knew Latin, and he was living in Holland at the time, so I can't explain it."

Malebranche provided Jacob with a copy of Descartes' *La Géométrie* in French, the original language, and Jacob opened it and began to study immediately. Although by this time Jacob's French was perfectly fluent, this was difficult reading. However, Jacob knew what he wanted, and he knew he was smart enough to master it. He read with quill, ink, and paper, working actively as he had learned to do with Pappus' works a few years earlier—Jacob knew that anyone who reads mathematics without quill and paper is not really serious about understanding it.

"*Monsieur* Bernoulli," Malebranche said the next afternoon, "you are probably finding Descartes difficult to read."

"Yes, it is difficult," Jacob answered, "but I think I can do it. Did you have trouble figuring it out for yourself?"

"Yes, I encountered the occasional road block, but through working at it seriously day after day and talking with other mathematicians here in Paris, I managed," Malebranche said.

"Well, then," Jacob said, "I guess I should be able to do that too."

"As you encounter difficulties, don't hesitate to ask me for help," Malebranche said. "I would hate to have you waste too much time on the basic concepts. That might not leave you enough time for the more interesting parts."

"Thank you, *Monsieur*," Jacob said. "So far I am doing all right."

"Did you know that Descartes deliberately made it difficult to read?" Malebranche asked.

"I wondered about that," Jacob said. "Do you know why he did it?"

"Well, I understand that he justified it in a couple of ways," Malebranche said. "First he said that he had given enough information so that anyone who had the proper background could figure it out. He thought that any more information would simply be redundant. He didn't want to insult his readers."

"I can accept that," Jacob said.

"Then," Malebranche continued, "Descartes said that he wanted his readers to have the genuine pleasure of completing his arguments. If we view mathematics as a sport, it would be inconsiderate of a mathematician to give it all away immediately. Mathematics is truly a treasure hunt—if someone tells you before you start where the treasure is hidden, it is no fun at all."

"You know, *Monsieur* Malebranche, that makes sense," Jacob said. "I guess I respect Descartes more after hearing that."

"Yes, but if what you want is to understand the mathematics so that you can pursue his ideas further, it would be futile to waste too much time on the foundations," Malebranche said. "I believe you are very serious about moving along in your studies of mathematics."

"Yes, indeed," Jacob said.

A little while later, Malebranche was sitting, looking at Bernoulli. Finally, he said, "Excuse me, *Monsieur* Bernoulli, wouldn't you be more comfortable using this footstool so that you can elevate your foot. It appears to be causing you serious pain."

"Well, *Monsieur* Malebranche, I appreciate the offer," Jacob said, "but my foot isn't bothering me too much. I don't want to impose. And besides, I believe you need the footstool more than I do." Malebranche had suffered from birth with a severe curvature of the spine, causing him persistent pain and limiting his mobility all his life. Jacob and Malebranche hadn't discussed it before, but Malebranche's suffering was obvious.

"I will ask the servant to bring us a second footstool," Malebranche said. "I have several. We are two diligent scholars who need whatever devices are available to help us in our pursuit of knowledge. Our study should not be hampered by physical pain any more than necessary."

Jacob struggled through *La Géométrie*, drawing sketches as needed, and finally comprehending the entire work, for the first time seeing algebra as the best way to study geometry. By this time he was adept at using an exponent to show a power of a variable and *x* (if not *y* and *z*!) for his unknowns. Descartes had thought it through carefully, producing brand new mathematics—what is sometimes now called analytic geometry—out of his own imagination. Jacob correctly saw it as the work of a genius.

Jacob and Malebranche also spent time discussing Descartes' philosophy—a topic that interested Malebranche (who was a priest in the Roman Catholic Church) far more than Jacob at this point. However, since Jacob recognized his debt to Malebranche in making the *Géométrie* available to him, he joined these discussions with apparent enthusiasm. His university studies in philosophy and theology had prepared him well for such debates. The two men discussed Descartes' famous statement *cogito ergo sum* [Latin for "I think, therefore I am"], and the difficulty of rationalizing Cartesian philosophy with the theology of the Church of Rome. Malebranche was convinced that Descartes' philosophy could be adapted to the teachings of the Catholic Church, even in the dispute over transubstantiation, although many Roman Catholic theologians found the Cartesians' approach too close to that of the hated Protestants'. Is the bread that is used in the Eucharist actually transformed into the body of Christ (the Roman Catholic view) or is it only a symbol of the body of Christ (the Protestant view)? A more fundamental question explores the relation between faith and reason.

While he was in Paris, Jacob also did some work on astronomy, another subject that his father had prohibited him from studying. The Latin motto that he had taken for himself—against my father's wishes I will study the stars—was true. He studied them in earnest. Using a borrowed telescope, he studied carefully the path of a comet in 1680. He concluded that a comet is not ephemeral—it doesn't appear for a brief time and then evaporate, as was the common belief at that time—and that a comet travels on a predictable path, orbiting the sun in the same way that the planets do, although often in a much larger orbit. His calculations convinced him that the 1680 comet should return on 17 May 1719. Whether or not that prediction was true (and his calculations were not correct as it turned out), then it is ludicrous to say that a comet is an omen of some calamity. Despite Jacob's mistake, he was correct about comets in general. A comet is not a fleeting sign from heaven indicating imminent misfortune. Therefore, Jacob said, it was foolish for people to make decisions based on that false reading of the heavens.

As he talked with people in Paris, however, he found that his radical view was not popular, so he decided to adjust it slightly. He then wrote that the head of the comet is not an omen—it cannot have anything to do with future events here on earth—but that he couldn't be absolutely certain that the tail does not indicate something. He announced that the tail is changeable and thus its shape might possibly have some significance. That appeased his critics without opening him up for criticism from scientists. It was a mild concession that protected him from attacks from all sides.

10
Jacob Travels to Holland and England

In 1681, Jacob traveled to Amsterdam, where he may have met with Jan Hudde (1628–1704), the foremost mathematician in Holland—in fact, the most important mathematician in all of Europe at the time. Hudde was a serious scholar of Descartes' mathematics and, using Descartes as his starting point, Hudde had devised two rules for dealing with polynomial equations that moved mathematics further toward the development of the calculus.

Hudde, who had worked extensively with his teacher Frans van Schooten, the translator and editor of the expanded version of Descartes' geometry, was a logical person for Jacob to meet. It was in Holland—not France—that serious mathematics was being pursued at the time.

"*Monsieur* Hudde," Jacob began. "No, I'm so sorry, Sir. In France I was careful to address people in French. In Holland I would like to use Dutch, but unfortunately I don't know the Dutch language. How should I address you, Sir?"

"The Dutch equivalent of *Monsieur* is *Meneer* (Mr.), but it doesn't matter," Hudde replied. "Perhaps it would be easier if we simply communicate in French, which I believe we both speak easily."

"No, no!" Jacob said in French. "At the very least I would like to address you correctly, *Meneer* Hudde. Was that right?"

"That was fine, *Herr* Bernoulli," Hudde said to Jacob in German. "German is your native language, isn't it?"

"That's right, although it is really not important," Jacob said. "Allow me please to begin again, *Meneer* Hudde. I notice that you boldly use a letter as a variable to represent any real number when you write mathematics, regardless of whether it stands for a positive or a negative quantity. Descartes didn't recognize negative numbers, as I remember. Isn't it risky to allow the variable to stand for a negative?"

"But it is essential," Hudde replied. "You see, Descartes, brilliant though he was, ignored negative numbers. Nonetheless, they are legitimate numbers. If algebra is to help us, we certainly need to be able to represent negative quantities with variables. Otherwise we lose at least half of the value of algebra. You have to admit that a debt is just as real as a credit in the world of business, and that is just one small illustration of negative quantities in mathematics."

"Yes, I suppose that is true," Jacob said.

"Allowing the variable to stand for both negative and positive quantities has not interfered with my work in the least, *Herr* Bernoulli," Hudde said, "and it has helped me dramatically. Furthermore, when we are solving an equation in algebra we frequently don't know whether a quantity will end up being positive or negative until we reach a solution (sometimes it even turns out to be positive sometimes and negative at other times!), so clearly the variable needs to cover both signs. Take a look at this." As Hudde showed Jacob his latest work, Jacob could see that the variables for negatives were indeed allowing him to do some important work.

"Do you mind if I read this through, *Meneer* Hudde?" Jacob asked, indicating the work in his hand which was written in Latin, a language they both could read and write easily.

"If you want to do that, *Herr* Bernoulli, that is not where you should start," Hudde gently corrected him. Walking to a table in the corner of the room and picking up another essay, he continued, "I would recommend that you begin with this essay that I wrote a year ago. Otherwise my more recent work will not be as clear as you would like. You need to follow my reasoning in the order that

I wrote it. Please feel free to sit down here and read it. Can I offer you a cup of tea?"

"That would be delightful, *Meneer* Hudde!" Jacob said. "Thank you so much!"

From Amsterdam, Jacob went on to the town of Leyden where he stayed for ten months, getting to know the mathematics professors there and perhaps teaching several classes for them. Since he would have lectured in Latin, his Dutch students would have been able to understand him perfectly.

From Holland, Jacob went on to London, where he was eager to meet John Flamsteed (1646–1719), the Astronomer Royal, who would soon move into and direct the new Royal Greenwich Observatory, in a position that Flamsteed would hold for the rest of his life. Jacob also met with Robert Boyle (1627–1691), familiarizing himself with that scientist's brilliant work in chemistry. Jacob learned how Boyle had discovered the fudamental difference between mixtures and compounds in chemistry, and he listened carefully to Boyle's description of his research into the chemistry of combustion and the process of respiration in animals, a subject that Jacob's nephew Daniel would study in his own doctoral dissertation 40 years later..

Jacob also talked with Robert Hooke (1635–1703), looking with fascination at his beautifully illustrated volume *Micrographia*, showing the world of things so small that they could not be seen with the naked eye. Hooke also described to Jacob an exciting new plan for a tubeless telescope, whose eyepiece was mounted separately from the lens so that the distance between the two could be changed as needed. When Jacob pressed him, Hooke admitted that the first tubeless telescope was actually not his own invention and that he hadn't yet constructed one himself. The first one had been made by the Italian lens maker Giuseppe Campani (1635–1715), but Hooke was eager to construct his own. As a superb contriver of things both mechanical and optical, this was well within Hooke's abilities.

In London Jacob learned of the mathematical writings of the English mathematicians John Wallis (1616–1703) and Isaac Barrow (1630–1677). In his 1669 textbook on geometry, Barrow had included information on the new work on maxima and minima—finding the greatest and the least possible value for an algebraic expression—and a useful technique for finding them. Barrow did not claim that this was his own original work, but his explanations were clear, involving the construction of the tangent to a curve (the straight line that hits the curve at only one point and that demonstrates the slope of the curve at that particular point), and Jacob studied that too. A few years later he would realize that Barrow's geometry is actually part of the foundation of the developing field that would later be known as the calculus.

By the time Jacob returned to Basel in 1681, he had mastered both Barrow's and Wallis' work, and he was almost up to date on all that was happening in the development of mathematics and science both in England and on the continent. His travels had allowed him to accomplish what he had set out to do. He boldly turned down an invitation to serve as a parish priest in Strasbourg, resolving instead to concentrate on mathematics back in Basel. He knew that his younger brother Johann, always his eager pupil, was ready to work with him as they put together Jacob's latest studies in mathematics.

11
Jacob Settles into Life in Basel to Lecture and Learn

Once again, Jacob's father was impatient for his oldest son to begin his career and accept a position as a pastor in the Reformed Church, but by this time Jacob had independently decided against that move. Realistically assessing his own abilities and goals, he was satisfied that turning down the offer from Strasbourg had been a good decision.

"Jacob," his father approached him with great concern, "do I understand that you have turned down that excellent position in the church in Strasbourg without consulting me?"

"Yes, Father," Jacob replied. "As you know, I have been working diligently on mathematics for several years, and I can't stop now. I am working at the forefront of mathematics today, and I must continue."

"Now, wait a minute, young man," his father said. "I cannot accept this. I was willing to let you travel after you completed your studies, but it was always clear that afterwards you would accept a position in the Church and make it your career. I know you understood that. Otherwise I would never have allowed you to go on your travels."

"I'm sorry, Father," Jacob began carefully. "I realize that you planned for me to devote my life to preaching the gospel, but instead I have found my own calling: mathematics. Like Martin Luther, I must follow my own calling. Perhaps you didn't know that Martin Luther's father had planned for him to become a lawyer. He must have been aghast when Luther instead chose a life in the Church. But you will recall that Martin Luther boldly said, 'Here I stand. I can't do anything else. God help me.' I say the same to you."

"No!" his father barked. "I have worked for years preparing you for your distinguished career in the Church. I feel as if you had just slapped me in the face."

"No, Father" Jacob said sadly. "This is not intended as an insult to you. I am truly sorry that you can't understand my passion for mathematics. You should know that when I am exploring mathematics, I do it with a near religious fervor—this is not a mere whim. This will be my life. Please don't condemn me for it."

"No! Religious fervor is for religion!" his father declared.

"If a man feels passionately about his vocation, it is his religion, Father," Jacob said. "I am a scholar. God has chosen me to pursue an

Jacob Bernoulli.

understanding of our world at a fundamental level. I have met with all the great mathematicians in the world, I have won their respect, and now I must join them in their work. Mathematics provides the foundation for all of science, and I must play my part. I have no intention of forsaking the Protestant religion, but that is not where I will make my career."

Jacob's father sighed. "You are a foolish young man. You are throwing away a brilliant career where you would have been respected universally. I can't believe it."

"No, Father," Jacob corrected him, "I am throwing nothing away. I fervently hope I will have a brilliant career, but it will be in mathematics, God willing. I have been working diligently toward this goal for several years, and I have no intention of stopping now."

In 1682 at the age of 28, Jacob decided to publish in the *Acta Eruditorum* [*Acts of the Scholars*], a scientific journal from Leipzig, Germany, his discoveries about comets and their orbits. He had discussed his research with scientists that he met on his travels, and now was the time to publish it.

At about the same time he published another article *De gravitate aetheris*, concerning the weight in the atmosphere of the aether, the mythical substance that many scientists of the time thought explained such phenomena as gravity. He wrote that it is obvious that air has some weight since we can measure atmospheric pressure with a barometer. He noted that he agreed with Malebranche, his host in Paris, who also doubted the existence of the aether, although neither Malebranche nor Jacob had a good alternative to explain the mysteries of the universe. In his article, Jacob argued for the wisdom of geometry and physics, which he thought between them were far more likely to produce a plausible explanation of the physical world than the mysterious aether. Certainly there was nothing more than circumstantial evidence for aether's existence anyway. Jacob considered this still a work in progress and he eagerly awaited the next development.

In 1683, Jacob presented himself to the citizens of Basel as a lecturer in physics, offering lectures on the experimental mechanics of both solid and liquid bodies. His brilliant lectures, which were marked by clarity and enthusiasm, quickly became so popular that Jacob was soon earning a significant amount of money from his teaching.

"Heinz, my friend, a hearty good morning to you!" 25-year-old Peter greeted his friend at the Basel city hall one morning. "Did you see this notice?"

"I was just looking at it," Heinz said. "That is a lecture that young Jacob Bernoulli is offering on mechanics and physics. It sounds most intriguing. He asks only two *Pfennig* for the lecture tomorrow evening, and I believe I will attend."

"What a good idea," Peter responded. "How would it be if I stop by your house tomorrow at five o'clock and we go together?"

"What fun that will be!" Heinz said. "I hear that *Herr* Bernoulli has been learning about many fascinating things on his journeys."

The following evening Peter and Heinz walked together to the community hall where Jacob would be speaking. When they arrived, they saw Jacob setting up his equipment in the front of the hall, testing his apparatus carefully to be sure that everything would function perfectly.

"Good evening, gentlemen," Jacob began when the crowd quieted down. "I am pleased to see so many of you for my first lecture. I am planning to do a series of five lectures, each on a different topic of mechanics. Before I begin, I have only one request: if you cannot hear me or if what I am saying is not clear, please interrupt me immediately. I will do my best to answer any questions you may have. When I finish, I hope you will all be willing to leave the two *Pfennig* that I have requested for your tuition on the table here at the front of the hall.

"My topic this evening," Jacob continued, "is capillary action. Are any of you familiar with the term?" Jacob scanned the audience and saw only looks of curiosity. "Capillary action is something that you all have witnessed. Consider this dry cloth that is touching this puddle of liquid. Look closely and you will see that the liquid is slowly seeping into the cloth. Isn't it odd that it can move across, not just down?

"Now, please observe this narrow glass tube which I have inserted vertically in this vessel of water," Jacob continued. "Notice that the liquid is rising in the tube—it is going up, not down. It is the same phenomenon: capillary action. Now consider for a moment, please, the quill that you sometimes dip in ink so that you can write on a document. What keeps the ink in your plume so that you can write several words between dips in the ink? It is the very same phenomenon: capillary action."

"Pardon me, *Herr* Bernoulli," one of his listeners called out, "could we please see that demonstration with the tube one more time?"

"Certainly," Jacob said, removing the tube from the water and shaking out the remaining liquid. "Now I insert the tube once again, holding it steadily upright. Do you see that the liquid is once again rising?"

"Thank you," his questioner said with satisfaction.

"Now there are several variables we need to consider," Jacob continued, as he explained about the difference in the quality of a liquid—oil or water or mercury—and his audience could see that the capillary action was different in the more viscous liquids. Then he proceeded to show them the effect of a wider tube as compared to a narrower tube, before he moved on to a scientific explanation of why it worked.

Next, he proceeded with an explanation of the measurement of barometric pressure and the way capillary action allows us to measure the pressure of the air in the atmosphere on a pool of mercury. He explained that the height of the column of mercury depended on the pressure exerted by the air on the pool of mercury in which the vertical tube was standing.

As he talked, there was an occasional gasp of wonder at a new revelation, but otherwise by now the lecture hall was silent. His audience did not want to miss a single trick. At the end of the demonstration, Jacob announced that he was indebted to Robert Hooke (1635–1703) of the Royal Society of London for parts of his demonstration. "I spent some time in London talking with Mr. Hooke," Jacob explained, "and he seemed pleased to show me some of the devices he has made. I have seen his famous book *Micrographia*, a beautiful volume with amazing drawings of the microscopic world." Jacob then explained further that in fact the first functioning barometer was built several years earlier by an Italian named Torricelli in 1643.

When the lecture was over, Jacob announced that his next lecture would be the following Thursday evening at the same time in the same place. His topic then would be the process of combustion—a topic that Robert Boyle (1627–1691) in England had done some fascinating work on. He explained that he had seen Boyle's demonstrations and had been amazed. He thought his listeners would have the same reaction. Everyone gladly left the money on the table for Jacob, and several asked if they might try Jacob's experiments with the glass tubes for themselves. Jacob supervised them carefully as they saw for themselves how capillary action works. "Look at this, Heinz," Peter said to his friend. "The water really is climbing up the tube! I wouldn't have thought it would be possible!"

"God in heaven! So it is!" Heinz agreed.

As they left, Peter thanked his friend warmly for encouraging him to attend the lecture. "Heinz, you were certainly correct that young Bernoulli's lecture would be fascinating. He has a real knack for explaining difficult things, and I think it would be safe to say that he has truly seen the world. I had never worried about why a rag absorbed water if it wasn't even submerged in the water, or why the ink stays in my plume as I write. This was fascinating!"

"I believe *Herr* Bernoulli has been studying with the most important scholars in Europe," Heinz said. "As he was talking, I was reminded of times when I was a schoolboy and learned something exciting and new. What could be more fun than that?"

This was popular education for ordinary people—people who were becoming aware that there were exciting developments in the world of science. It turned out that many people were willing to pay for the privilege of hearing a knowledgeable scientist speak, particularly once they realized that they could understand what he was saying. Jacob's father may have finally admitted to himself at this time that perhaps Jacob's mathematical and scientific studies had not been so foolish after all.

In 1684 at the age of 30, Jacob married Judith Stupanus, who, like Jacob, had grown up in Basel. She was the daughter of a successful businessman in town. Jacob and Judith had two children—a son named Nicolaus (after his grandfather) and a daughter named Verena—but neither of these children chose to study mathematics or physics. Jacob's son Nicolaus became a painter like his uncle Nicolaus, and his daughter married a successful businessman.

In 1687 Jacob devised a method for dividing a scalene triangle into four equal parts geometrically with a pair of perpendicular lines. His friend Jean Christophe Fatio-de-Duillier from Geneva had sent this challenge to Jacob after learning of it from the esteemed Dutch mathematician Huygens, and Jacob was able to accomplish it through a remarkably skillful manipulation of Descartes' geometry. Jacob was pleased to publish this result and with it to win further respect from the scientific community.

Also in 1687, four years after his return to Basel, 33-year-old Jacob finally was chosen for the chair of mathematics at the university in Basel. Now he would be recognized as Professor of Mathematics. The long years of standing up to his father's pressure to make the move into his "real" career in the Church had finally paid off. However, as a professor, his salary was almost as small as his father had predicted many years earlier.

For the rest of his life, Jacob had to supplement his salary through private tutoring in addition to the fees he earned from his popular extracurricular physics lectures. As a serious professor of the latest mathematics, he was soon attracting students to Basel from throughout Europe. A succession of these students, who had heard of Jacob's reputation as a brilliant teacher, lodged with the Bernoullis, paying for their professor's hospitality as well as his tutelage. This was an additional burden for his wife Judith as well, but they were both committed to Jacob's career.

Beginning in 1690, Jacob's lectures in physics and mechanics were listed in the university catalog, with an official meeting time on Thursday afternoons at 3:00. By this time, his lectures had become so popular that their location had to be changed—there wasn't enough room for all the eager listeners in the original location. Now they had to be conducted in the dining hall of a music school nearby—a sure indication of Jacob's success.

One of the students who may have lodged at Jacob and Judith Bernoulli's house at this time was a poor but very bright young man named Paul Euler (pronounced *oiler*). Euler's brilliant son Leonhard would later become an extraordinary student of Jacob's younger brother Johann. Paul Euler was preparing for a career in the Church, but he was intrigued by what he had learned in mathematics, and he eagerly studied with Jacob Bernoulli. At the time, Paul Euler also came to know Jacob's brother Johann, since they were about the same age, may have lived for a time under the same roof, and frequently listened to the same lectures. In 1688, Paul Euler was a successful respondent to Jacob in a series of disputations on ratios and proportions. From that time on, the connection between the Eulers and the Bernoullis was always close.

In the early 1690s, Jacob Bernoulli discovered a revolutionary way to graph a point or an equation on the plane in a way that was totally different from Descartes' method. His brother Johann, who was now in his early twenties, was eager to hear all about it. At this time, they often worked together on mathematics—Johann didn't want to miss anything that his brother found interesting.

"Now, Johann," Jacob began, "take a look at this new method of graphing that I have just come up with."

"But I thought Cartesian graphing did everything we would need to do," Johann protested. "I've never had trouble with it."

"I think my new method is even better than Descartes'," Jacob said. "From the origin, I'm going to draw a ray going off to the right, technically forever, although we would never draw it that way."

"So far it sounds just like Cartesian graphing to me," Johann complained.

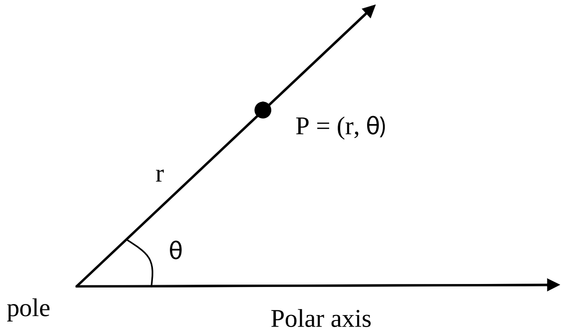
"Let me continue!" Jacob protested. "You may think you know everything, but I still have some advantages over you. Hear me out! What I have discovered is truly revolutionary."

"All right," Johann said meekly, "I'll listen."

"Okay, so we have this ray with one endpoint at what Descartes called the origin," Jacob said. "The new way to locate a point is by taking two measures: the angle away from that first ray measured in terms of π, and then the linear distance out the ray."

"Aha!" Johann said, suddenly seeing where his brother was going with this. "So if we want to graph a point on the Cartesian y-axis above the origin, we would describe the 90° angle as $\pi/2$, since π would be 180°—aren't we saying that 2π would be one complete rotation? Then we would measure how many units away from the origin you need to go to reach the point."

"That's right," Jacob said. "Then the ordered pair of that location in polar coordinates would be $(\pi/2, 3)$ if the point is three units above the origin. So the first entry of the ordered pair gives the angular measure, and the second gives the distance. That's all we need to know."



Graphing with polar coordinates.

"Yes, and that would be the same as the point (0, 3) in Cartesian coordinates," Johann said. "I think I like this. Is it original with you? Did you make it up yourself?"

"I devised it myself, and as far as I know I am the first person to do it," Jacob said. "I'm in the process of writing it up for an article in *Acta Eruditorum*."

"What advantages do you think this new method has over Descartes' method?" Johann asked.

"Well, for one thing," Jacob said, "it allows us to consider motion—not just things that are stationary. For another, we could represent a given point by indicating that it had been rotated 1/4 of the way around, or we could describe that same point as 9/4 or even 17/4. I am finding it remarkably useful."

"I like it, Jacob," Johann said. "Do you mind if I try using it in my work?"

"No, I'd actually like you to use it," Jacob said enthusiastically. "Please tell me if you discover anything more about it. You truly are my best critic, and I value your opinion."

Jacob published his method of polar coordinates in the *Acta* in 1691, and it created a major sensation among the mathematicians of Europe. In fact, Johann was later surprised to learn that Jacob was not the first person to use it. Isaac Newton had come up with the same scheme several years earlier, but, typically for Newton, he didn't publish it until 40 years later in 1736. Since Jacob published it first, he is the one who deserves credit for it.

At this time, the two Bernoulli brothers were happily working together on their mathematics. Jacob had started ahead of Johann, but Johann always grasped the new concepts so quickly that they were truly operating as equals in most ways most of the time. Although they fought brutally when they were older, that strife was still in the future at this time. They were still two congenial adventurers blazing new paths into the wilderness, making many exciting new discoveries as they worked. It wouldn't have been half as much fun without an accomplice.

In addition to his explorations into mathematics in 1691, Jacob foolishly involved himself in university politics.

"Professor Schmidt," Jacob addressed one of his colleagues one afternoon, "what do you think of our policy of allowing a professor to teach in a field that is not his area of expertise?"

"Well, Professor Bernoulli," his colleague replied, "I must admit that I have never worried about it. We have many fine scholars on our faculty."

"But think about it, Sir," Jacob persisted. "Our students come to the university to learn from scholars in their fields. A professor of law who claims to be a scholar of Greek is a fraud as I see it!"

"Now, wait a minute," Professor Schmidt corrected him. "That is the way the university has always functioned. I don't think we need to disturb the workings of the university in an attempt to be purists. In general, you must agree that our program is excellent."

"But don't we want to present the best scholarship that we can to our capable young students?" Jacob asked. "Shouldn't all of us be the best scholars we can possibly be?"

"I must say that I would not want to make an issue of it," Professor Schmidt admitted. "Wouldn't it be better for you to simply concentrate on doing the best you can in your own field?"

Jacob, ignoring the advice of his older colleague, spoke to others on the faculty and even to some officials of the university, causing considerable unpleasantness, and eventually resulting in a suspension of his position on the faculty. He was justifiably perceived as trying to stir up unrest. Whether or not Jacob was right, those in positions of power found the current arrangement thoroughly satisfactory and were unwilling to see the system that served them so well turned upside down.

Fortunately for Jacob, his father had enough prestige in the community to step in and push the authorities to reinstate Jacob. In fact, Jacob may have been disingenuous in his stance since when he had been trying to join the faculty as a professor only a few years earlier, he had proposed twenty theses to defend in a wide variety of fields (not just mathematics). "Would anyone like to hire me as a professor of moral philosophy?" Jacob had asked. In later years, future Bernoullis (his nephews and great-nephews) were also guilty of this same "offense." Although Jacob was probably right that the university policy was not ideal for academia, it was not a battle that he was going to win for many reasons, and the policy certainly helped more than one Bernoulli over the years.

12
Leibniz's Calculus vs. Newton's Fluxions

Twenty-five years earlier, in the years 1665–1666 and far from Basel, the 23-year-old scientist Isaac Newton was a refugee from Cambridge University on his family's farm in central England, far from the highly contagious disease called the plague, which had forced the closing of the university until the danger passed. Newton spent those 18 months thinking and discovering and experimenting, in what has since been called his *Anno Mirabilis* [Miraculous Year]—the months when he made more brilliant discoveries in science than perhaps any other single person has ever done in so short a time. His only restrictions were the limits imposed by his own imagination and curiosity—and these were amazingly vast and deep.

Newton had retired to the country where he was at liberty to think and experiment and pursue his discoveries wherever they might lead him, free from any cares. Looking back on that time, the older Newton said, "In those days, I was in the prime of my age for invention and minded mathematics and [natural] philosophy [meaning science] more than at any time since." Newton was so consumed with his research that he did little else during those months, often forgetting to bathe and sometimes even to eat or sleep. There is a story that his cat grew luxuriously fat from eating all the untouched food that was set out for the possessed young scientist.

It was during this time that Newton came up with the basic concepts for the part of mathematics that he called fluxions and that we now call the calculus. Newton's brilliant insight in mathematics was to see that further exploration of algebra and geometry must center on motion. He saw a curve not as a collection of points, as in Euclid's classical geometry or Descartes' analytic geometry, but rather as movement and change. His fluxions (related to the words fluent and flowing) were a dynamic study, in which he looked at the instantaneous speed of a particle and the area found under its curving path. He did that by constructing the ratio of the distance covered in the journey to the time it took to cover it. His method was to look at both those measures as they were reduced to the smallest possible increment—not zero, but very, very close to zero—what is now called the infinitesimal or the infinitely small.

Archimedes (287–212 B.C.) had approached the infinitesimal more than 1800 years earlier, as had many others since then, but there is no doubt that Newton was the first to see how to use it to solve a wide range of problems. However, he would have been the first to admit that his new explorations in mathematics were not an isolated piece of work carried out by him alone—he was one actor in a continuum of scientific discoveries. By this time, he was already well grounded in the mathematics that had been discovered over the preceding centuries, and the field was ripe for further development. He said once that he had accomplished all that he had because he was able to stand on the shoulders of giants—of Archimedes, Huygens, Descartes, Wallis, etc. The mathematical world was ready for the discovery of the calculus, and Newton was the first to put it all together. Because of Newton's amazing accomplishments in mathematics, he is today considered one of the four greatest mathematicians of all time, coming after Archimedes and before Leonhard Euler (1707–1783) and Carl Friedrich Gauss (1777–1855).

Newton's discoveries during those 18 months were not limited to mathematics alone. Stories tell of his experiments in optics and vision. Once he looked straight at the sun for as long as he could stand it in order to understand the way the eye works, although he paid for that experiment later as he was forced to spend several painful days inside a dark room while his eyes returned to normal. In fact, he was lucky that he didn't blind himself completely at that time. Another time he experimented with the effects on his vision of varying the shape of his eyeball by inserting what he called a bodkin [a large, blunt needle] as far behind his eyeball as he could in order to observe the effects on his sight of a change in the curvature of his retina. That experiment could also have had disastrous results, but once again the ingenious scientist escaped unharmed.

At this time, Newton also experimented with refracting a ray of light into the spectrum of colors from violet to magenta using a prism, although people argued at the time that this was nonsense. It was common knowledge that normal daylight, which is clearly white, couldn't possibly be composed of all those colors! While Newton knew he was right about this and his other discoveries, he felt no strong desire to convince anyone else of that. He was apparently content to explore solely for the sake of exploration. It was relatively unimportant to him what use others would make of his discoveries.

There is another famous story (which is probably not literally true) describing Newton's inspiration at watching an apple fall to the ground, leading to his discovery of the universal law of gravity. Many of his contemporaries were critical of Newton's concept of gravity, since he couldn't explain why it worked. They enjoyed mocking his concept of a certain "drawing-ness"—as they dubbed the power of the attraction that he called gravity—between the earth and the sun or between an apple and the earth. The skeptics condemned it as farfetched, but later scientists discovered that gravity actually does work the way Newton said, regardless of its cause. In later years when Newton was hailed as a hero, he modestly claimed that he had merely been like a boy who had happened to find some particularly pretty stones while playing carelessly on the seashore.

By the time Newton reached middle age, he enjoyed the passionate respect of his colleagues and the general public throughout England. He was knighted and called Sir Isaac Newton, served for a time in parliament, became an effective and energetic Master of the Mint, and was buried in a place of honor in Westminster Abbey. Although he was not of noble birth, by the time he was 30 years old he was esteemed as the most noble scientist in the English-speaking world. While he had few friends and generally worked in splendid isolation, his genius was universally recognized in his homeland.

The story of Gottfried Leibniz is totally different, although, like Newton, he certainly was a genius. Unlike Newton, he was far more than just a scientist—he was a polymath. He was a savant who worked brilliantly in many fields, from law to philosophy to mathematics, and who, unlike Newton, also enjoyed communicating with others. Newton enjoyed his own company far more than the company of others.

Leibniz's greatest regret was that he was not a nobleman by birth, one who could enjoy the privilege of making witty conversation in the courts of Europe throughout his life just because of who he was. He would have loved to devote his time to any intellectual pastime that he chose for as long as he wished, as Newton had been able to do throughout his adult life.

In spite of his remarkable accomplishments, however, Leibniz was never awarded the status of a nobleman as Newton was. When Leibniz died, he was buried in an unmarked grave, unrecognized and unsung, with only his former secretary in attendance at the interment. The contrast with Newton seems grossly unfair.

Leibniz's father had been a professor of philosophy who possessed a large library in which young Gottfried was allowed to read widely after his father's death. Having an insatiable curiosity, the boy began doing this from a very young age. Although he attended the local school beginning at the age of seven and was instructed there in Latin and Greek, he had already taught himself those languages in his desire to read all the books in his father's library. There was no stopping that child! Leibniz went on to study at the university in Leipzig, successfully completing his Bachelor's degree at the age of 16 and his master's degree in philosophy a year later.

However, when Leibniz applied for his doctorate in law, the university at Leipzig refused, perhaps because they considered him too young, but more probably because they were limited in the number of doctorates they could award in a given year. They apparently reasoned that since Leibniz was only 20 years old, he could certainly wait another year for his doctorate.

Not one to accept defeat, Leibniz promptly traveled to the nearby university at Altdorf where he submitted his brilliant dissertation and was soon granted his doctorate in law there at the age of 21. Clearly a gifted and accomplished student, he was immediately offered a professorship in law at Altdorf, but he promptly turned that down. A provincial university was too small a setting for him.

Leibniz then attached himself to a series of noblemen who appreciated and were eager to exploit his brilliance. As an expert in the law, Leibniz had much to offer, and he was pleased to make himself valuable to noble sponsors at the same time that he saw the world. In this way, he was able to take part in the world of nobility, even though he was not personally a member of that club. While he worked for one of these noblemen on a diplomatic mission in Paris, Leibniz was delighted to discover the world of mathematics beyond rudimentary reckoning. Studying seriously under the guidance of the Dutchman Christian Huygens, the most important mathematician of his time, Leibniz was enthralled, and with his genius he was able to progress rapidly.

He then traveled to London, also on a diplomatic mission, and there he met with members of the Royal Academy, in his free time demonstrating his brilliant plans for a calculating machine that he claimed would add, subtract, multiply, divide, and take square roots. It was an inspired idea, although the many delays in the actual construction of the machine caused him considerable embarrassment with his London contacts over the years. In fact, the machine never functioned as he had planned.

While Leibniz was in London on two separate trips, he heard of Newton, although he did not actually meet him. It is possible that at the time he saw at least one privately printed piece of Newton's work on fluxions, but, if he did, there is no record of that event. With his limited mathematical background at the time, he probably could not have understood Newton's writings even if he had had the opportunity to study them carefully.

When Leibniz later returned to the continent and explored mathematics in his occasional free moment, he began to see the need for the analysis that might be possible with some new mathematical tools. He then wrote to Newton asking for some information on his work, which Newton eventually sent to him, although he encrypted it so thoroughly that Leibniz was able to learn nothing from it. Newton's goal was to establish his priority without actually divulging anything. Since Leibniz couldn't decipher it, neither of them gained anything from that correspondence.

Seven years after Newton's discovery of his fluxions, Leibniz once again found himself on a diplomatic mission in Paris between 1672 and 1676. It was during those years that he discovered his calculus. Leibniz's creativity was stunning. He devised the notation that we use today: $f(x)$ and dx and $\int$, compared to Newton's fluxions, which used such symbols as x with one dot above it or x with two dots above it for the first and second derivatives. Since Leibniz's calculus is the version that we use today, mathematicians prefer Leibniz's calculus notation, although that may be simply because it is familiar to us.

Leibniz's calculus came to be called the calculus as a shortened version of the title of his 1684 article in *Acta Eruditorum,* the scientific journal that Leibniz had helped to found and where Jacob Bernoulli published his first discoveries. The Latin title of Leibniz's article was *Nova Methodus pro Maximis et Minimis, itemque Tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illis* calculi *genus* ["A new method for Maxima and Minima as well as Tangents, which is neither hindered by fractional nor irrational quantities, and a remarkable type of *Calculus* for them"]. The word *calculus*, found in the word *calculi* at the end of that title, is the Latin word for *pebble*, referring to the pebbles in an abacus used for calculation. The word fluxion is seen today as no more than a quaint reference to Newton's system that did not win out. Today we study only Leibniz's calculus.

Newton did not publish anything about his fluxions until 1687, many years after his own discovery of it and three years after Leibniz's first publication of his calculus. Even then, Newton's method of fluxions was only an incidental part of his monumental work *Philosophiae Naturalis Principia Mathematica* [*Mathematical Principles of Natural Philosophy*] which is usually called simply *The Principia*. In it he used his fluxion method occasionally, although most of his proofs used only traditional geometry. The work includes no clear presentation of Newton's method of fluxions.

Unlike Newton, Leibniz eagerly published his calculus, but his first article on the calculus in *Acta Eruditorum* was a mere six pages of dense and exotic calculations with very little explanation. During his youthful travels, Jacob Bernoulli had heard something of Gottfried Leibniz, an impressive scholar in mathematics and many other fields and one of the founders of the journal *Acta Eruditorum*, which Jacob now read regularly. Since Jacob had already mastered all that Huygens and Descartes and the other great mathematicians of Europe had presented, Jacob knew that it was now time to read Leibniz.

After studying Leibniz's article carefully, Jacob Bernoulli described it as an enigma rather than an explanation. Other savants, who had also tried to read it, had simply given up. Leibniz's work used discoveries that he had made sometime before 1677, at least seven years earlier than the current article. Presumably this article was an improvement on that earlier work, but it was still unclear to even his most determined reader—Jacob Bernoulli. Although Jacob wrote to Leibniz in 1687, asking for some clarification, he received no answer, probably because Leibniz was traveling for his noble sponsor and thus was out of touch with his correspondence. He would find his mail when he returned home six months later. Thus, Jacob had no choice but to persevere on his own. In later years, Leibniz admitted that his calculus was as much the work of Jacob and Johann Bernoulli as it was his own. Although Leibniz certainly had the initial inspiration, he needed the Bernoullis to present it to the scientific world.

In 1691 Jacob published two essays on Leibniz's infinitesimal calculus, based on his teaching of the subject to his private students at the university in Basel and the work he and his brother Johann had done together. These essays were the first presentation of the infinitesimal calculus that were clear enough to allow other mathematicians to begin to comprehend the subject. The development of Leibniz's calculus from the end of the seventeenth century into its many forms was the major accomplishment of the eighteenth century, due in great part to the revolutionary work of the Bernoullis and then of Leonhard Euler, the brilliant son of Jacob's former student, Paul Euler.

Twenty years after Leibniz published his discovery, a war ensued over who should get credit for the development of the analysis that we call the calculus, the branch of mathematics that first allowed us to find the instantaneous velocity of a particle and the area contained within a curve. The priority battles continued long after the deaths of Newton and Leibniz, effectively cutting off English mathematicians from continental mathematicians. As a result, mathematics developed separately in England and on the continent for at least a century. In great part because of the work of Jacob Bernoulli, his brother Johann, the Bernoulli brothers' students, and Leonhard Euler, the continental scholars were able to carry their investigations much further and much faster than their English counterparts, with the result that the English lost any competitive edge they might have had. Because Leibniz's mathematics was the active medium for the development of later mathematics, Newton's notation and English mathematicians lost out. That is the reason that fluxions are no longer a part of standard mathematics.

The war of the two calculuses is complicated in many ways. Newton certainly developed the analysis first but didn't publish it until much later. Leibniz may have seen some suggestions of what Newton had done when he was in London, but Leibniz's method was original, and he published it several years before Newton. The Newtonians who accused Leibniz of plagiarizing were wrong. On the other side, some continental mathematicians boldly accused Newton of plagiarism, saying that Newton could have read Leibniz's calculus, which had been in print before he presented his own. Both Leibniz and Newton are now respected as having independently discovered the calculus. Neither of them was guilty of plagiarism.

13
Johann Bernoulli Grows Up

In 1683, with Jacob lecturing in Basel on topics in physics but before he was named professor, his father decided that it was time to make preparations for his younger son Johann's career. He had lost the career battle with Jacob, but now he had another chance. Johann, a remarkably bright young man, had completed the standard schooling, and his father decided that with his keen mind the ideal career for him was in business. He saw Johann not as a brooding young man like Jacob, but rather as one with a quick wit and the perfect personality for a life in business.

"Johann," his father said to him one morning, "I have arranged an apprenticeship for you with a very successful businessman I know in the town of Neuchâtel. He expects you to arrive next Monday to begin your work with him."

"What did you say I am going to do?" Johann asked in horror. "Am I supposed to become a businessman?"

"Yes," his father said. "You are a capable young man who learns new things easily, and I believe you have the perfect personality for this career."

"But I wanted to study at the university, like Jacob," Johann protested.

"No, Johann," his father said, "you and Jacob are very different people. You have a keen and penetrating mind, and so a career in business is right for you."

"You may be right that I'm smart enough for this, but what if I don't want to become a businessman?" Johann asked.

"You will accept this apprenticeship," his father announced. "I would recommend that you take a little time this weekend and review your French studies so that you will be able to communicate with your host easily when you arrive."

It soon became clear that Johann had even less interest in becoming a businessman than his older brother had had in becoming a pastor. One year later, after many protests, Johann was finally allowed to return home to Basel to study at the university there. His father had lost once again. In 1685 at the age of 18, Johann stood in a debate against his brother Jacob at the university, with the result that Johann was granted the degree of Master of Arts so that he might begin the study of medicine, his father's second choice of a career for him. In 1690 at the age of 23, Johann passed the licentiate in medicine with a thesis on fermentation, a decidedly mathematical piece of medical research.

After he published this work, Johann quietly broke off his study of medicine for a few years. Mathematics was his interest, and he pursued it eagerly. He was determined to learn whatever mathematics his brother Jacob had learned, and soon they were operating at the same level, although Jacob later described Johann scornfully as his student, who, as Jacob had predicted many years earlier, would never be able to do anything in mathematics unless Jacob chose to teach it to him. In fact, that is not the way it happened, as their relationship became complicated in several ways. Nevertheless, they were both formidable mathematicians whom the mathematical world quickly came to respect, even when they showed little respect for each other.

Jacob and Johann, who were both working hard at understanding Leibniz's writing on the calculus, were the first people to genuinely understand the details and the potential breadth of its applications. Although Leibniz had discovered it and used it in a limited way, and Leibniz's friend Ehrenfried Walther von Tschirnhaus

(1651–1708) had explored some of its applications, neither Leibniz nor von Tschirnhaus had developed a clear presentation of the material, and neither of them had been able to generalize the techniques. While Leibniz and von Tschirnhaus had consistently limited themselves to solving specific problems, the Bernoullis could see that the work was important in a much broader way.

Jacob and Johann carefully read all the works of Leibniz and von Tschirnhaus that were published in the *Acta Eruditorum* between 1682 and 1686.

"Johann, look at this paragraph from Leibniz," Jacob said one afternoon. "He is looking for a good way to find the slope of a curve at a specific point."

"But Jacob, isn't that what Barrow did?" Johann asked. "Barrow was able to find the slope of the line using Descartes' geometry."

"Yes, Barrow did that," Jacob said, "but it's possible that Leibniz is taking this much further or maybe in a different direction. Let's reserve our judgment until we see where Leibniz is going with this."

Several minutes later Johann observed, "Leibniz certainly didn't worry about making it particularly clear."

"No, and I've written to him for clarification, but so far he has not responded," Jacob said. "However, we're smart enough. We should be able to figure this out. If Leibniz could do it, so can we."

Since the Bernoulli brothers knew nothing of Newton's work (which had not been published), their only source was Leibniz. As they continued working, they found that in fact Leibniz had taken it much further and deeper than Barrow had, and the more they worked the more enthusiastic they became.

"Yes, Johann," Jacob said a few days later, "I think Leibniz has done something completely original here. I don't think Barrow could have done this."

"No," Johann said, "I think you are right. I think Leibniz has an entirely new method of analysis that we can use in some fascinating new ways. This is exciting!"

Without any help from Leibniz, the two brothers deduced how his calculus worked, and they were amazed at how powerful a tool it was. Jacob was soon teaching the calculus to his private students at Basel, and both brothers were moving ahead with their researches. They both understood that they were standing at the beginning of a brand new field of mathematics, and they were eager to move ahead with it

Johann spent most of the year 1691 in Geneva, teaching differential calculus to Jacob's friend J. C. Fatio-de-Duillier at the same time that he worked seriously at deepening his own understanding of it. Several years later, Fatio's younger brother Nicolas would play an active role in the debate between the Leibniz camp and the Newton camp as they struggled to establish who deserved credit for the first discovery of the calculus. Leibniz's primary defender turned out to be Johann Bernoulli, who fought for Leibniz's side energetically for many years. For Johann, defending Leibniz's calculus was a crusade that must not be lost, while the English, with help from Nicolas Fatio, were similarly fervent. It is unlikely that Johann's tutoring of the older Fatio played a role in this dispute.

At that age of 24 in the fall of 1691, Johann went on from Geneva to Paris, where he was able to enter into serious mathematical discussions with Jacob's friend Malebranche and his circle of friends, who were eager to learn more about Leibniz's calculus. Johann was also in contact at the time with the Dutch mathematician Huygens, who had been in the forefront of mathematics as it evolved from Descartes' analytic geometry and who had served as Leibniz's first mathematical mentor while he was in Paris. Through his correspondence with the Bernoulli brothers, Huygens eventually became convinced that Leibniz's calculus was correct and important, although he did not use it as enthusiastically as the younger, more active Bernoullis did.

Johann's most important new contact in Paris was the Marquis de l'Hôpital, a member of Malebranche's group of mathematicians. L'Hôpital was recognized at the time as the greatest mathematician in France. The marquis was eager to learn the calculus, and Johann agreed to teach him for a very large fee, but under an agreement granting the marquis sole rights to the material. Since Johann's father was still reluctant to support his rebellious younger son, Johann welcomed this arrangement, which was to continue for several years, although he later regretted signing over his rights to the presentation of the calculus.

At first, Johann instructed the marquis in person both in Paris and at his country estate outside of Paris, but later the instruction continued by mail at l'Hôpital's request. Johann kept good records of his instruction, retaining copies of the letters he wrote to l'Hôpital in the years after he had left Paris. Several years later, when l'Hôpital surprised Johann by publishing a textbook on differential calculus, *Analyse des infiniment petits* [*Analysis of the Infinitely Small*], Johann was pleased at first. L'Hôpital mentioned Johann's name on the title page, but that was the only credit that Johann got. Whose work was it? Johann was astonished that his student had had this in mind!

Many years later, when Johann protested his rights to the calculus textbook, he had proof that in fact he was the author, not l'Hôpital. However, their agreement had been to give l'Hôpital free use of the materials, and so the textbook is still officially called l'Hôpital's book, and l'Hôpital's Rule on simplifying an expression which involves a fraction that has a zero in both the numerator and the denominator retains his name as well. Nevertheless, credit for the first complete explanation of the calculus as well as l'Hôpital's Rule should belong to Johann Bernoulli.

While he was in Paris at this time, Johann also met Jacob's friend Pierre de Varignon. Although Johann also taught Varignon the calculus, he did it not as a formal tutor but rather as a friend and colleague. Ultimately they developed a warm friendship, as evidenced by regular correspondence that continued for many years. This time there were no payments for instruction and no transfer of rights to the material.

In 1693, Johann began to correspond frequently with Leibniz, exploring with him the general principles of the calculus. Johann was to keep up his correspondence with Leibniz for many years, often keeping Leibniz informed of his brother Jacob's work as well. Jacob observed more than once that he was too lazy to be a good correspondent, although he also had serious health problems that interfered with his activities for much of his life. However, Jacob came to resent that correspondence.

"Johann," Jacob complained one day, "is that a letter from Leibniz to you?"

"That's right," Johann said. "I had written to ask him about that problem we were working on a couple of weeks ago, and he has just responded."

"I ask you, Johann, do you consider Leibniz your exclusive friend? You wouldn't have known anything about him if I hadn't introduced you, and I resent being left out of your communications. Leibniz must believe that in corresponding with you he is in communication with both of us, but that begs the question."

"Oh, well, I just wanted to get his reaction to what we are doing," Johann explained.

"But you didn't include me in that communication, did you?" Jacob asked.

"Well, no, but I assumed that if you wanted to communicate with him, you would write a letter yourself," Johann said.

"I resent your attitude. From now on, I would like to see letters you send to Leibniz, and I would like to read his responses. You owe me no less than this."

"Why don't you write to him yourself?" Johann asked as he stormed out of the room. "I'm not your secretary."

At this time, Johann was also regularly submitting articles both to the *Acta Eruditorum* (the journal Leibniz had founded) and to the

Journal des Sçavants, another scientific journal, which had been published in French since 1665. As a result of his writing, Johann was increasingly being regarded as a serious mathematician and not just as his brother's student.

In 1694 at the age of 27, Johann finally completed his doctoral thesis in medicine on the functioning of muscles in the human body—a decidedly mathematical exploration of a medical topic. Although it was a doctorate in medicine, since that was what his father demanded, there was no doubt that Johann was a mathematician. Ten days after completing his doctorate, he married Dorothea Falkner, the daughter of one of the city deputies. Johann was ready to begin his career, preferably in a comfortable position at the university in Basel. Unfortunately for Johann, however, the chair in mathematics was already inconveniently filled by his brother Jacob.

Instead, Johann reluctantly accepted a position as engineer for the city of Basel. The job was neither interesting to Johann nor well paid. As a result, Johann was desperate to find something else.

One evening about this time, Johann's wife Dorothea said, "Johann, I'm so proud of you. You are now a recognized scholar: Dr. Bernoulli!"

"Yes, Dorothea, it took me awhile to get to this point," Johann admitted, "and I have to admit that it's a little unclear where I should go from here. Certainly I have no desire to continue in this position of city engineer. If only Jacob were not sitting in the only chair in mathematics in Basel!"

"Well, he is older than you," Dorothea observed, "by 13 years."

"Actually, it's more like 12 1/2 years," Johann said. "He was born at the end of December."

"At this point, I don't see that it makes much difference," Dorothea said. "You are both grown men and impressive scholars, but Jacob got a head start on you. So what are you thinking of doing?"

"Huygens is exploring finding me a chair in Holland at one of the universities there," Johann said. "What would you think about moving to Holland?"

"I've never thought about it," Dorothea said. "I've always lived in Basel, and, of course, I would prefer to remain at home. I had always assumed we would live in Basel and bring up our children here. I love Basel."

"But since we don't have a proper income, Basel is difficult for us," Johann said. "I'm sure we would be able to find people in Holland who speak German or at least French, and I really don't think we have much choice. For me, a career in Basel is closed unless I want to follow one of my father's plans and enter the world of business or become a medical doctor. Obviously, I don't want to do either of those things."

"Then let's see what Huygens is able to find," Dorothea suggested. "I guess I'm prepared to do whatever you think is best."

In 1695, with help from Huygens, Johann was called to be professor of mathematics at Groningen, a major university in Holland. Twenty-eight years old and with the esteemed title of professor, Johann was officially now the equal of his brother Jacob. In that secure position he could pursue his career as a mathematician. He traveled with his wife Dorothea and their 7-month old son Nicolaus to Groningen, where Johann taught mathematics and physics successfully for ten years. It certainly would not have been Dorothea's first choice, but she was pleased that Johann's career was beginning well, and she found that Groningen was a beautiful city. As a tribute to Johann, the city of Groningen now has a square that is called Bernoulli Square.

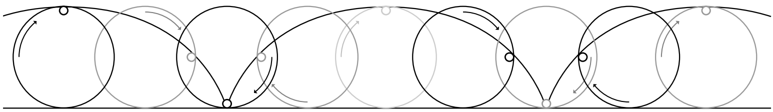
Like his older brother Jacob, Johann was also a brilliant teacher. In his writing as well as his teaching, Johann demonstrated that he not only understood the calculus in all its details—he could also explain it to those who were not already conversant in the field. He was passionate about the calculus, and he was able to inspire in his students a similar passion for the field. He is probably responsible more than anyone else for the triumph of Leibnizian differential and integral calculus over Newton's method of fluxions, which in fact accomplished the same thing.

In later years, when he described the battle over who deserved credit for the discovery of the calculus—Leibniz or Newton—Johann referred to his boyhood study of Livy's *History of Rome*. In Book II, Livy described the scene where Horatio Cocles bravely chose to defend the bridge over the River Tiber against the approaching Etruscan army. Although all but two of his comrades had fled, Horatio stood his ground and defended the bridge against the invading forces. Johann saw himself as Horatio, bravely defending Leibnizian calculus from the arrogant, misguided English. The comparison was not lost on his readers—any educated person would have known the story of Horatio and would have understood Johann's point.

14
Two Curves Studied by the Bernoullis: The Isochrone and the Catenary

In 1659, several years before either Leibniz or Newton had discovered the calculus, Huygens had been able to establish that the isochrone (from the Greek *iso* meaning same and *chrone* meaning time) was the curve along which an object under the influence of gravity would reach its lowest point in the same amount of time from any point on the curve. The curve was in fact an already familiar curve known to mathematicians as the cycloid, a curve that is formed by the path of a marked point on a wheel, as the wheel rotates along a level path. Huygens had used the isochrone in his invention of an accurate pendulum clock, using that curve as the path of his pendulum bob.

In 1690 at the age of 36, Jacob Bernoulli was so bold as to use Leibniz's calculus—what he called for the first time "the integral calculus"—to derive the same equation in an essay in the *Acta Eruditorum*, validating Huygens' result and demonstrating the usefulness of the calculus. Leibniz read and approved of Jacob's work and adopted Jacob's name *integral* for that part of the calculus. The mathematics community acknowledged Jacob's demonstration of this effective use of the calculus to solve an existing problem in mathematics as a significant accomplishment both for the calculus and for Jacob himself.



The cycloid: The path of one point on the wheel as it turns.

Having succeeded with the isochrone, Professor Jacob Bernoulli proposed a new problem in *Acta Eruditorum*, asking his readers to find the equation of the catenary curve—the curve traced by a flexible chain that is suspended from both ends and allowed to simply hang between those points. Today we see the catenary in the curve of the giant cables that support a suspension bridge. Galileo (1564–1642), who had studied that curve, incorrectly guessed that it was probably a parabola. Galileo was unable to calculate the equation of that curve because it can be done only with the use of the calculus, which had not yet been discovered in Galileo's time. When Jacob proposed the problem, he also had not yet found the equation of the catenary curve, but he still assumed, like Galileo, that it must be some kind of parabola.

In June of 1691 Leibniz, Huygens, and Jacob's then 24-year-old brother Johann (who was still living at home in Basel at the time) discovered the equation—which was not a parabola—and published it in the *Acta Eruditorum*. Jacob was mortified. His little brother, who had not yet completed his doctoral thesis at the time and who was eager to establish his own mathematical reputation, had beaten him! However, any damage Johann's discovery might cause to his brother's ego was of no concern to Johann. From this time on, Jacob and Johann's warm relationship deteriorated rapidly.

In 1718—27 years later and 13 years after Jacob's death—Johann, who was then 51 years old, was still crowing about his phenomenal success. In a letter to his friend Pierre Rémond de Monmort (1678–1719), a French mathematician with whom Johann exchanged many letters over the years, Johann described the scene in the Bernoulli family the morning after Johann discovered the equation of the curve in 1691. Johann boasted in his letter:

> Consider this, *Monsieur*! I am going to astonish you. I am telling you that my brother, try though he might, could not discover the equation of the curve of the catenary. Why should I be modest? I can tell you, *Monsieur* de Monmort, that I am the Bernoulli who discovered what that curve is. I proved that it is not a parabola, and, I assure you that Leibniz did not give me any hints. The discovery belongs to me, and I will prove it to you. You say that since my brother posed the problem, then it must be his property, but I say no. He may have posed the problem, but he couldn't solve it! Isn't that pitiful? I have to admit that at first neither of us could solve it, and we suffered. It was incredibly difficult. After all, even the genius Galileo couldn't do it.
>
> However, when *Monsieur* Leibniz announced in the *Acta Eruditorum* that he had solved it without divulging what the solution was, then the challenge was even greater. I must admit that I was awake one whole night working on this—remember my brother had been working on it for months and months without success. I have to tell you that I was suddenly fortunate—the solution came to me in a flash at the end of my long night of searching! When my brother arose the next morning I was able to present my solution to him. Poor soul, he was still miserable in his ignorance!
>
> "Stop! Stop! Jacob don't frustrate yourself any longer!" I said to him. "Don't torment yourself anymore, Jacob! It isn't a parabola, so you will never be able to make it fit the equation of the parabola. I have the solution, which I am delighted to share with you. Look at this!"
>
> Please believe me, *Monsieur*! My brother didn't have a clue what the curve was! If he had known, he would certainly have announced it to everyone. He would certainly not have allowed me to publish my result before him if he had had a choice.

> Later in my brother's correspondence with Leibniz, he indicated to Leibniz that we had solved the curve. Ha! It wasn't we—it was I! I am the Bernoulli who was awake that entire night and finally succeeded. At first Leibniz didn't know which of us had done it, but my brother and I certainly knew. Later Leibniz learned that the solution was mine, not my brother's. *Monsieur* de l'Hôpital has seen the evidence, and he agrees.

The catenary is a unique curve. Its equation involves hyperbolic geometry, a part of trigonometry that originates from the curve of the hyperbola. When the parabola is centered at the origin in Cartesian coordinates, it can be easily represented by the equation $y^2 = 4ax$ where a is the focus and x and y are the variables. The hyperbola is a basic part of algebra, while hyperbolic geometry cannot be approached without the calculus.

The catenary curve is steep at the two ends since at those points the total weight of the chain is heaviest. Toward the middle, the curve becomes less and less steep because the total weight there is decreasing where it approaches its lowest point at the curve's center of gravity. When the Bernoullis and Leibniz struggled with the curve, they were dealing with it mainly as a curiosity, but since then it has been critical to physics and engineering. The Gateway Arch in St. Louis, Missouri, is stable because it is an inverted catenary that is 630 feet high and 630 feet wide at its base. Johann used the integral calculus to reach his solution—without the calculus, the solution would have been impossible—so once again the calculus was proving its usefulness.

Eventually, Jacob too found the equation of the catenary, and he was able to present a more complete solution. Although Jacob's discovery of the catenary curve was both original and brilliant, Johann, who was eager to be seen as a scholar separate from his brother, refused to acknowledge its validity until 1715, long after Jacob's death. He would not give Jacob that satisfaction. In 1701 Johann presented his solution to the Paris Academy through his friend Varignon, but as it was presented, the solution was not complete. Jacob did not hesitate to use it as an occasion to ridicule his brother in print.

Relations between Jacob and Johann never improved after this time. They were both belligerent competitors, each determined to best the other at every opportunity. It was pointless, of course, and wasted much of the Bernoulli mathematical genius.

15
More Mathematical Challenges from the Bernoullis

In 1696 Johann, who was then a 29-year-old professor of mathematics, had settled into life in Groningen. His career was going well, he was far from his principal rival—his older brother—and he was happy.

"Well, Dorothea," he said to his wife as he returned home one evening, "I have to admit that my teaching is going very well. I am confident in the subject matter, and my students seem to be ready to learn. It is more fun than I expected it to be."

"Yes, at first you seemed a little pessimistic about the move to Holland," Dorothea said.

"Well, you were too, if I'm not mistaken," Johann said. "But isn't it fun to watch little Nicolaus as he grows?"

"He seems to be a very bright little boy," Dorothea said.

"Yes," Johann agreed. "I can hardly wait to see how he develops."

"Do you suppose he will want to study mathematics?" Dorothea asked.

"Oh, I don't know," Johann said. "Not everyone should be a mathematician, you know."

"No, not everyone," Dorothea said, "but wouldn't it be fun if he chose to do it?"

"I'm not so sure about that," Johann said.

At that time, the esteemed professor of mathematics at Groningen proposed a problem from his university office to be printed in *Acta Eruditorum*. Johann called his problem the *brachistochrone* from the Greek *brachistos* meaning shortest and *chrone* meaning time. The problem called for the discovery of the equation of the curve of quickest descent under the influence of gravity between two given points, one higher than the other and not on the same vertical line. It would be the path that a lazy hawk, wishing to coast as quickly as possible from one point at the top of a tree to a lower point on a nearby tree would fly on. Although Johann gave his readers until the end of 1696 as the deadline for entries, by the time the deadline arrived, only one correct solution—from Leibniz—had been submitted besides Johann's own.

"*Herr* Bernoulli," Leibniz replied to Johann's letter in the first few days of 1697, "I am certain that there should be more solutions to your challenge. Would you consider sending it out again, this time in the form of a pamphlet directed to the mathematicians who would have a chance of solving it? You might then extend your deadline, perhaps until Easter of this year?"

"Yes, I could do that," Johann agreed in his next letter. Then he added, "Whom do you think I should send it to?"

"Well," Leibniz responded by the next post, "certainly to your brother. And what about the Marquis de l'Hôpital? You have been working with him for several years. Do you believe his calculus is ready for that challenge?"

In his response, Johann expressed some doubts about l'Hôpital. "The marquis has learned a great deal, but I'm not sure he is ready for this challenge. However, I might as well send it to him too anyway. He would probably be offended if he knew we had omitted him from our list."

"How about Newton?" Leibniz wrote.

"Newton?" Johann wrote. "Do you think he actually has a calculus that he could use to solve my problem? I've begun to wonder if his method even approaches the usefulness of your calculus."

"We would be very foolish to underestimate Sir Isaac Newton," Leibniz warned. "The English claim that he has come up with a technique very similar to ours, and I have no reason to doubt them. Why don't you send it to him and see what happens?"

On advice from Leibniz, Johann extended the deadline until Easter of 1697. Johann sent personal copies of the challenge to each of the three other men, addressing his pamphlet to "the most brilliant mathematicians in the world." He explained that the curve was well known to geometers, and he stated clearly that, although one might wish that the correct solution might be a straight line, it was not.

The group of brilliant mathematicians was indeed a select group, and four of them were adequate to the challenge on their own. Leibniz had already solved it with little difficulty the day he received it. Jacob soon solved it as well, but with a totally different approach than Johann had used, and certainly without any help from his brother.

Newton, who had not been an active mathematician for many years, found the pamphlet in the mail when he arrived home from a long day at the London mint. He stayed up until 4 o'clock the following morning, solving it successfully using his own method of fluxions. It was a good puzzle, and Newton was not going to sleep until he had conquered it. He felt no need to announce that he had lost most of a night's sleep in solving it—his niece, who served as his housekeeper, later provided that information—although in his place Johann might have been tempted to mention that. When Newton submitted his solution, he did so anonymously. All the other entries were signed.

Although l'Hôpital, the other man on the list, wanted very much to solve the problem, he was unable to accomplish that by himself—he asked for and received generous help from Johann, whom he was still paying handsomely for his tutoring services. With Johann's guidance, l'Hôpital finally succeeded, hoping to keep his name on the list of the most brilliant mathematicians in the world.

In both *Acta Eruditorum* and the pamphlet, Johann had described the problem as a new one that he was inviting mathematicians to solve. He noted that the reward was neither gold nor silver. Instead, any successful solvers would have the supreme satisfaction and the profound respect that came to those accomplishing a great intellectual feat—a prize far more valuable than a mere financial reward.

When Johann opened the entries on Easter day of that year, he looked at Leibniz's, l'Hôpital's, and his brother's before opening the envelope from England. Although it was anonymous, Newton was the only English mathematician to whom Johann had sent the problem and he was the only person in England who Johann thought had a chance of solving the problem. Despite his prejudices, when Johann studied Newton's anonymous entry, he recognized the correct solution at once and observed, "I can tell the lion by his claw"—a comment indicative of Johann's opinion of Newton. Although Newton's notation was different from that of the continental mathematicians, it was certainly correct. In May of 1697, Johann published all five solutions (including his own) in the *Acta Eruditorum*.

Not surprisingly, the brachistochrone provided another forum for the strife between Johann and Jacob. Johann's solution involved restating the mechanical problem as an optical one—one that he could solve using Fermat's (Pierre de Fermat 1601–1665) principle of least time. Through that insight, he discovered that like the isochrone (see Chapter 14), the equation of the cycloid was the solution. Johann's was a brilliant solution to the brachistochrone, showing remarkable perception, but it offered nothing for mathematics in general. It simply solved the specific problem at hand.

In contrast, his brother Jacob constructed a much more involved argument, considering the big picture rather than the individual problem, and coming up with what turned out to be a new field of calculus—the calculation of variations—in the process. Jacob's solution was similar to Leibniz's.

Over time, historians of mathematics have concluded that both Johann and Jacob were brilliant but radically different mathematicians. Johann's wit was sharp and quick—his agile mind allowed him to see through a problem quickly and arrive at brilliant conclusions. Jacob, by contrast, operated more slowly and often came up with deeper and more general solutions.

In 1697, Jacob, who was then 43 years old, proposed another problem which he called the isoperimetric problem (from the Greek *iso* meaning same and *perimetric* meaning distance around), which asks for the determination of the curve of a given length between two points that will enclose the maximum area.

This classic question of calculus has roots in Greek and Roman mythology. In the *Aeneid*, Virgil tells the story of Princess Dido, who announced that she wished to buy land to build a city for her people on the northern coast of Africa. In reply, the wily King Jambas told her that he would sell her as much land as she could enclose in the hide of a bull, thinking that she was sure to be disappointed and would be forced to give up her plans. Dido, who turned out to be wilier than the king, had the skin cut up into narrow ribbons which were then sewn together end to end. She was able to expand the area even further by attaching the two ends of her long ribbon to two points on the seashore some distance apart so that her perimeter was even bigger, giving her the area of a half circle for her city. She cleverly solved the isoperimetric problem and was able to build her now famous city of Carthage.

When Jacob proposed the problem, it was already well known that the circle gives the maximum area if one doesn't have the advantage of Dido's stretch along the seashore. In primitive cultures around the world, circular houses have always been the favorite plan because they make the most efficient use of building materials. The problem was to prove it mathematically. The shape was easy; the proof was remarkably difficult.

Both Johann and Jacob published their solutions to Jacob's isoperimetric problem in 1701. Both men used the calculus to solve it with completely different approaches, but their end results were much the same. This time viewers from outside the family might have supposed that both Bernoullis won, but, since neither brother came out ahead, both were disappointed. Neither of them felt that he had won the contest.

The letters that Jacob and Johann exchanged at this time are full of arguments about methods of approach to several problems, pursuing the open warfare that the two brothers would engage in for the rest of their lives. This was not so much a struggle to discover new mathematics cooperatively as it was a contest to demonstrate who was more clever and more important. Over time, it appears that Johann was the one who pushed this strife more often, although there was certainly fault on both sides.

"Johann," Dorothea greeted her husband one evening as he returned home from the university at Groningen, "you have a letter here from your brother."

"Blast him anyway!" Johann exploded.

"But Johann," Dorothea remonstrated, "he is your brother. Why do you constantly fight him?"

"Because he is a lout!" Johann said. "He is a good mathematician, I admit that, but he has tried over and over to belittle my work. Why does he pursue me like that?"

"Well, perhaps you should read the letter before you get too worked up about it," Dorothea suggested. Johann tore it open and stood reading it, getting angrier and angrier.

"Yes, he's trying to minimize another one of my discoveries! I tell you, Dorothea, I hate him!" Johann said. "He likes to explain that he is the one who taught me the calculus, and therefore I can't do anything original myself. Yes, he taught me the first parts of calculus, but as time went on we worked together, each of us helping the other. He couldn't have done all that he has done without my help."

"Then why can't you work together like that again?" Dorothea asked.

"Because my brother is a rat!" Johann said, and strode out of the room.

16
Jacob Bernoulli's Mathematics

Jacob Bernoulli spent much of his life refining and expanding the calculus, which presents many occasions to contemplate the infinite. As an innovative scientist, Jacob boldly grappled with it at many levels. His poem on the paradoxes of infinity (written in Latin and translated by Martin Mattmüller, Basel, 2009) is a fine example:

Ut non-finitam seriem finita coërcet
Summula, et in nullo limite limes adest:
Sic modico immensi vestigia Numinis haerent
Corpore, et angusto limite limes abest.
Cernere in immenso parvum, dic, quanta voluptas!
In parvo immensum cernere, quanta, Deum!

Just as a finite sum confines an infinite series
And in what has no bounds there's still a bound,
So traces of divine immensity adhere to bodies
Of lowly kind, whose narrow bounds yet have no bound.
What a delight to spot the small in vast expanses,
To spot in smallness, what a joy, the immense God!

Although the significance of the calculus often eludes those who use it—the calculus can often feel more like a set of mechanical algorithms than a mind-boggling creation—Jacob didn't forget to stand back and admire the amazing mathematical machine that he was developing. He was an artist with a grand view of his vast subject.

Jacob Bernoulli's other great work—in fact his only book—is his *Ars Conjectandi* [*The Art of Conjecturing*], which was not published until seven years after his death. Jacob worked on it for 25 years off and on between 1680 and his death in 1705. He commented more than once that he suffered from both laziness and illnesses, and that both interfered with finishing his great work. In fact, the book represents a tremendous amount of original, creative work. The final product, *Ars Conjectandi,* which was eventually published in 1713, is the first complete study of the science of probability.

Jacob's first published work on that subject was a challenge that he proposed in the *Journal des Sçavants* in 1685 when he was 31 years old, two years before he was named professor of mathematics at Basel. It was written in French and directed to both intellectuals and recreational players of games. His question concerned a fictional game in which players A and B take turns throwing one die or number cube. He outlined two possible sets of rules, in both of which the winner is defined as the first player to throw a one on a standard die. With the first set of rules, in round one player A throws the die once, and then B throws it once. In the second round, player A throws twice and B throws twice. In the third round, A throws three times and B throws three times, etc. The other set of rules begins with A throwing once, then B throws twice, then A throws three times, then B throws four times, etc., continuing in this way until one of the players throws a one. The question is what is the ratio of player A's likelihood of winning to player B's likelihood of winning? Another version of this problem, later called the St. Petersburg Paradox (see Chapter 27), arose several years later when Johann's son Daniel and his cousin Nicolaus, son of Johann and Jacob's artist brother Nicolaus, were pursuing mathematics.

Unlike his article in *Journal des Sçavants*, Jacob's book *Ars Conjectandi* was not intended just for recreational mathematicians. It was published in Latin, the language of serious scientists. This time he was addressing his colleagues and students at the university of Basel as well as scientists throughout Europe—the people who read

Acta Eruditorum and who wanted to keep up on the latest developments in science. Jacob may have hoped that the book would appeal to aristocrats, too, with its obvious application to games of chance which the leisure classes had time to enjoy, but they would have to make the effort to read it in Latin.

Whereas Gerolamo Cardano (1501–1576), (the mathematician who had first attempted to teach the blind to read) wrote on probability, his work wasn't published until after his death in 1576, and it was largely ignored in the development of the theory of probability. The next serious study of probability appeared a century later in the 1650s in the correspondence of Blaise Pascal (1623–1662) and Pierre de Fermat (1601–1665), although neither of them published their results.

In 1657, Christiaan Huygens (1629–1695), the Dutch inventor of the first accurate pendulum clock and Leibniz's mentor in Paris, published his findings in his book *De Ratiociniis in Ludo Aleae* [*Concerning the Calculation of Games of Chance*]. In it, Huygens gave a method for calculating how many times a pair of dice should be thrown in order to make the probability of a given outcome worth the risk to the individual player of betting on the game. Huygens assumed that a player might say to himself: "As a rational person, I would like to know my chances of winning before I commit myself to playing." Jacob used Huygens' work as he began work on his own study.

In his *Ars Conjectandi*, Jacob presented the study of probability as an attempt at quantifying the likelihood of an event so that one could take risks intelligently. It was general knowledge in the 1680s, for example, that an insurance policy would be a poor investment unless one knew what the chances of a given event were, although at this time it was unclear how one could figure those chances.

Jacob realized that before the event, short of fixing the game there is generally no way of absolutely guaranteeing what an outcome will be. As a result, he began his study by exploring similar situations in which he could study the results *a posteriori* [after the fact] of a known event, in the hope that he might be able to predict *a priori* [before the fact] what was likely to happen in a similar situation in the future with what he called "moral certainty"—a benchmark that he would set. For this he looked back to Aristotle, whom he had read seriously in his undergraduate studies in philosophy. Aristotle had recognized that since absolute certainty is usually unattainable, an intelligent person should set a standard of certainty beyond a reasonable doubt.

A probability of one, or 100%, before an event happens is usually impossible. If the probability that it will rain is 0.5, that means that it is equally likely to rain or not rain. If the probability that it will rain is 0.9, it is much more likely to rain than not to rain, whereas if the probability of rain is 0.15, rain is unlikely. Jacob chose his standard for "moral certainty" of an event as a probability of at least 0.999. If an event's probability was 0.999—i.e., that it would happen 999 times out of 1,000—he said he could safely predict that it would happen. That was the closest he expected to come to a guarantee. After the publication of his book, his standard of "moral certainty" was soon adopted by mathematicians throughout the world.

The first section of Jacob's work is basically a summary of Huygens' book, with Jacob's own commentary bringing it up to his time. Section two summarizes the work that had appeared more recently on such topics as permutations (the number of arrangements that are possible for a certain number of events) and combinations (the number of possible sets of a certain number of events if two events occurring in different orders were considered equivalent). Section three explores the uses of probability in games of chance, and section four explores how probability can be applied to practical matters (such as the calculations of insurance premiums), moral questions (such as deciding when it is safe to conclude that a person who has been missing for several years is dead), and civic issues (for example, the construction of laws within a modern society).

When Jacob wrote about games of chance, following the example of Huygens and Pascal, he assumed that both players had an equal chance of winning—mathematicians considered those the only fair games. If two players were mismatched, it was the duty of the player with the greater probability of winning to give himself a handicap so that the chances would once again be even.

Jacob decided to study mortality statistics (the age at which specific subjects died) of a given population for which he could get the statistics, and he would use those data to calculate the mortality rate of a similar population in the future. As a result he could use that calculation to predict the likelihood of a similar individual dying at a certain age with moral certainty.

If it is likely that a person will die by the age of 40, then an insurance policy that is written for a person who is already 39 years old would need to be very expensive, while a policy for a person at age 18 would be less expensive, since he is expected to survive much longer. By the time the 18-year-old reaches the age of 40, if he has continued with his policy and paid his premiums for many years, he has already paid for his own generous payoff. It is safe to assume that the insurance company would prefer not to pay out any more money than necessary, but unless an intelligent policyholder has some chance of collecting, he would not be interested in buying a policy at all. Jacob recognized that probability does not guarantee that a given person will live to a certain age, but for the whole population it is reasonably accurate.

Jacob knew that the more statistics he could study, the more reliable his predictions would be. He decided that he could estimate the probability of a given event to any degree of accuracy he wished, using what he called the Law of Large Numbers, which appears in the last part of his *Ars Conjectandi*. While he recognized that these predictions provided only general statistics, he argued that he could make valid predictions about the population in general, and that would allow an insurance company to charge a reasonable fee for a policy.

Jacob's limited correspondence with Leibniz shows a search that Jacob pursued for many years as he struggled to complete section four of his book. In order to construct his arguments, he needed hard data, and such data were difficult to find. He repeatedly begged Leibniz to send him a copy of Johann De Witt's work, which spelled out mortality statistics from a study in Holland in 1671. He knew that Leibniz had read it, and he was convinced that Leibniz still had a copy of it. Jacob wanted it, believing that those statistics would allow him to conclude his Law of Large Numbers.

Jacob Bernoulli's gravestone in the cloister of the Basel Münster: Translated from the Latin: Jacob Bernoulli, incomparable mathematician.

Although eventually Leibniz responded to the letter, Leibniz did not send the work, saying that he no longer had it. At the time, Leibniz was still traveling extensively, researching the lineage of the House of Brunswick, his sponsor's family, as he tried to produce the history of that family. He probably didn't have the data with him as he traveled, and they may or may not have been still in his possession at home in Hannover. Without Leibniz's data, Jacob's proof was incomplete and he had no hope of finding such statistics anywhere else. In pain and fatigue, he set the manuscript aside one last time before he died.

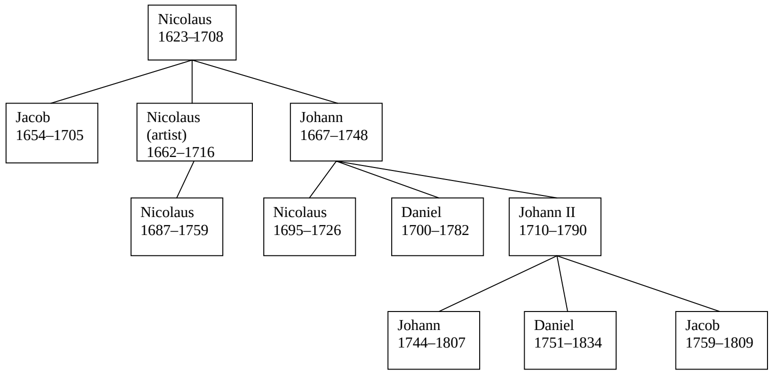
At his death at the age of 51 in 1705, Jacob's book *Ars Conjectandi* was still not complete, but he left clear directions as to what should be done with it. Above all, he directed that it should not be put in the hands of his rival—his brother Johann—even though it might have seemed reasonable to an objective observer that the mathematical brother of a great mathematician was the logical person to see the book through to publication. Instead, Jacob directed that neither Johann nor any son that Johann might produce should even catch a glimpse of his work before it was published.

He requested that his own artistic son Nicolaus (some scholars have suggested that he designated his brother Nicolaus' son Nicolaus instead), should take the manuscript with him to Paris (where he planned to study painting) and show it to Jacob's friend Varignon who could decide if and how it should be published. Apparently Nicolaus did not do as his father asked, although he did make contact with Varignon. Jacob's wife and son held onto the manuscript and eventually turned it over to Thurneysen Brothers, publishers in Basel, and it was finally published to great acclaim in 1713, eight years after Jacob's death.

17
Johann Bernoulli Returns to Basel with His Family

Between 1695 and 1705, Johann Bernoulli and his wife Dorothea thrived in Groningen, Holland, producing several children and firmly establishing Johann's scholarly career. In 1699 both brothers Jacob and Johann were elected to the Paris Academy, and in 1701 Johann was elected to the Berlin Academy, to which Jacob had been elected several years earlier. It was clear to the academic world that the Bernoulli brothers, like Leibniz and Newton in the generation before them, were the most important mathematicians of their generation.

In 1700, Johann and Dorothea's second son Daniel, who was to become a mathematician and physicist as renowned as his father and his uncle Jacob, was born in Groningen. When Daniel Falkner, Dorothea's ailing father, realized that his grandchildren, including his namesake Daniel, were rapidly growing up far from where he could see them, he began to pressure Johann to move back to Basel. The entire Bernoulli family over several generations continued to feel a remarkably persistent tie to Basel—a tie that Johann must have felt as well—and in 1705 Johann and his family finally acceded to his father-in-law's wishes and moved back to Basel. Johann had recently been offered positions at the universities in both Utrecht and Leiden, two of the finest Dutch universities, where Johann might have preferred to relocate if the choice had been only up to him, but he had to turn those offers down.



The Bernoulli family's mathematicians 1600–1850.

Unfortunately, there was no position in mathematics available for Johann at the university in Basel when he moved his family back home, although he must have been aware that his brother Jacob's health was rapidly deteriorating. As it happened, Jacob died almost as soon as Johann and his family arrived back in Basel, and Johann quickly applied for and was offered his brother's chair. Given their stormy relationship over the previous five years, this might not have pleased Jacob. Nevertheless, the university at Basel considered itself lucky to claim once again Europe's greatest mathematician as its own. Basel was not at the center of the intellectual world, but thanks to the Bernoullis it continued to earn universal respect for its mathematics.

Johann and Jacob's father was still alive in 1705 when Johann, who was then 38 years old, returned to Basel, and it seems reasonable to assume that by this time he had accepted his two mathematical sons' careers. His other sons were doing well also. Nicolaus, who was between Jacob and Johann in age, was respected as a painter, and his youngest son Hieronymus was carrying on the family business in spices.

Certainly it appeared that both Jacob and Johann had achieved the ultimate in mathematical prestige, and through a combination of their salaries as professors, their private tutoring, and renting rooms to foreign students who needed a place to stay while studying at the university, both mathematicians prospered. Contrary to their father's worries, they were not a strain on the family resources. Although Johann suffered occasionally from gout like his brother Jacob (who had been only 51 years old when he died), Johann was much healthier and lived into his eighties, extending for many years the Bernoulli monopoly of the chair of mathematics at Basel—a monopoly that would be extended yet again by his sons and grandsons.

Like his older brother, Johann Bernoulli was renowned as a brilliant lecturer in mathematics. Although he considered basic instruction in algebra little more than an annoyance and he avoided such teaching assignments whenever he could, students had high respect for the clarity of his lectures at all levels. While he was by no means rich, Johann himself was known to quietly pay the tuition for a student whom he considered worthy but for whom the cost of tuition was too high.

Johann Bernoulli and his wife Dorothea had five sons, of whom three, following the family tradition, became mathematicians. The youngest two had successful careers in business. Johann and Dorothea's four daughters, two of whom died in infancy, were expected to marry advantageously within Europe's merchant class if they were so lucky as to survive childhood. The first daughter Anna Catharina, was born in 1697 and died only a few months later, distressing her father greatly. Their second daughter, also named Anna Catharina, was born less than a year after her sister. She had a happy childhood, often playing and working with her brothers Nicolaus and Daniel, and was to grow up and marry well. Anna, as her family called her, would survive her first husband, marry again, and end up surviving her second husband as well. A younger daughter named Dorothea also prospered and remarried after the death of her first husband, this time to a pastor who later became professor of Hebrew studies at the university in Basel.

Nicolaus, Johann and Dorothea's first son and apparently Johann's favorite, was born in Basel before the family's move to Groningen, Holland. Quite naturally, Johann instructed his oldest son in mathematics from an early age since the boy seemed fascinated by it and learned it easily. At the age of eight, Nicolaus already spoke Dutch, German, French, and Latin fluently. He was clearly a very bright child, and his father Johann enigmatically chose a career in law for him.

"Father," young Nicolaus asked his father one day during the period when the family was still living in Holland, "what is the best language to speak? Which language is most important?"

"Oh, those are not easy questions," Johann said. "It really depends on what you intend to do with the language. Living in Holland, Dutch is certainly important to us."

"Obviously, Father," Nicolaus agreed, "but if we are not just talking about day-to-day use of language, then which language is most important?"

"Once again, there is no easy answer," Johann said. "As a scientist, I must speak and write Latin every day."

"Okay," Nicolaus said, "I can do that."

"Yes, you can," his father agreed, "although you will need to perfect it further as you grow older. However, Latin is not the end of my answer. In the world of modern Europe, I think most people would agree that French is the language of choice for people who are well educated. Anyone who does not speak French well will certainly be viewed as an ignoramus."

"But I can speak French," Nicolaus said, "so I'm no ignoramus."

"No, but you still need more practice with it," Johann said. "You have never actually lived in the French language, and we will have to arrange for you to do that before you are ready to go out into the world. You never really know a language until you have lived in it."

"Okay," Nicolaus said, "but what about German? We speak German at home, but is that just because you and Mother grew up speaking German? When I do arithmetic, I always do it in Dutch. I think Dutch may be the most useful language for me. You do arithmetic in Dutch, don't you, Father?"

"Of course not!" his father said. "I always do arithmetic in German."

"I wonder why you do that," Nicolaus said. "Do you suppose I will need to learn more languages? Are there other languages that educated people need to speak?"

"Yes, there certainly are," his father said, "but you will have to wait and see what other languages you will need. It is possible that you will want to travel to England, and that means you may need to learn to speak English, too. But for now, I think you should concentrate on improving your mastery of the four languages you already know."

By the time Nicolaus was 13 years old, the family had returned to Basel, and he soon entered the university there, carrying on his studies in both German and Latin. He passed the master's examination in law at the age of 16, and then completed the requirements for the licentiate in legal studies in 1715 at the age of 20. However, like his father and uncle, mathematics continued to be his real love.

Johann and Dorothea's second son, Daniel, was born in Groningen in 1700. At that time, Nicolaus was five years old and their sister Anna was two. After the family's return to Basel in 1705, Daniel, who was then five years old, entered school there. Both in Groningen and in Basel, Daniel learned mathematics from his brother Nicolaus. By this time the two boys had become very close, and, as their uncle Jacob and their father Johann had done when they were younger, they worked happily together. They made a happy pair as they explored that abstract subject together. Sometimes they worked in earnest silence; other times the roars of laughter as they attempted to capture one more difficult concept could be heard throughout the house. Fortunately, unlike Jacob and Johann a generation earlier, this warm relationship continued into their adult years.

"Anna," the children's mother Dorothea asked her older daughter one afternoon, "what are you children playing with so earnestly?"

"Oh, you don't need to worry about that, Mother," Anna explained. "It's just some games with mathematics. Nicolaus is teaching Daniel how to do it, and it's such fun!"

"I'm not so sure that is the best thing for a young lady to be learning," her mother said with some concern. "You might be better off spending that time practicing the piano."

"Oh, no, Mother, I'm sure it's all right," Anna assured her.

"You do seem to be having a good time together," her mother said. "I guess there's no harm in it, and you are making good progress at the piano too."

18
Johann Bernoulli's Son Daniel Grows Up

Many years later, Daniel wrote to his friend Christian Goldbach (a family friend whom Daniel would come to know well when he lived in St. Petersburg):

> My brother Nicolaus became a mathematician almost accidentally. Perhaps it was because it came so easily to him that he didn't realize what astounding progress he had made in mathematics. He wanted to instruct me in the calculus that our father had taught him as well as what he had figured out for himself, although when we began studying together I was only 11 years old. He used all his talent as he tried to teach me—his inadequate little brother. Only after I had learned it did he realize that in the process of teaching me he had truly mastered both the differential and integral calculus completely. In fact, I believe that his plan for us was that we should discover mathematics together—he never saw himself as my teacher, although that is precisely what he was. At that time he assumed that I was as accomplished a mathematician as he was! Foolish Nicolaus! I was nothing more than his ignorant apprentice who had been able to master a few small pieces of mathematics with a great deal of help from him.

When Johann, their father, realized that his oldest son had been teaching young Daniel mathematics, he decided to test Daniel's progress. "Daniel, come here, boy!"

"Yes, Sir?" Daniel replied uneasily.

"I understand your brother has been teaching you a bit of mathematics," Johann said. "See if you can solve this problem," which he wrote quickly on a piece of paper. Daniel was delighted to accept the challenge and happily took the problem to his room. He was pleased that his father was finally taking an interest in his efforts at mathematics. He quickly solved the problem, which was not difficult for him, and immediately brought it proudly back to his father.

"Father," Daniel said happily, "here is the solution to your problem."

"What took you so long?" Johann demanded. "You should have been able to solve that while you were standing here! I thought Nicolaus said you were good at mathematics. Bah! You'll never achieve anything important. What a pity!"

Daniel was devastated. He had always suspected that his father preferred Nicolaus, but he had dared to hope that this time he might have accomplished something that his father would find worthy. Alas, that was not to happen. Any mathematics that Daniel would do in Basel would be with help from his brother Nicolaus or on his own, always with less than no support from his father.

"Daniel," his mother Dorothea asked a little later when she found him quietly reading by himself, "what's the matter?"

"It's nothing, Mother," he replied.

"I heard your father saying something to you, and I think it made you unhappy," she persisted.

"Oh, no, Mother," he said, "You don't need to worry. I have been learning a little mathematics from Nicolaus, but I haven't gotten very far."

"But you like mathematics too, don't you?" his mother asked.

"Yes, I like it, but I'm nowhere near as smart as Nicolaus," Daniel said.

"I don't think that is the case at all," his mother said. "I think you are every bit as bright as your big brother. Since he is older, of course he is further advanced in mathematics, but I'm sure that with time you will learn it as well."

"I'm not so sure about that, Mother," Daniel said, "but I guess I'll try to do a little more mathematics with Nicolaus. Maybe I can do something."

"I hope you will, Daniel," his mother said. "I believe you will succeed."

A family friend, Condorcet, explained many years later that the family obtained the honor of Daniel's brilliant work in science in spite of itself. This was an honor that the family had no right to claim since the family (with the exception of his brother Nicolaus and possibly his mother) did nothing to help Daniel. What Daniel accomplished, he did because of his own passion and genius.

Daniel completed his first degree at the university in Basel in 1715 when he was 15 years old and completed his master's degree in 1716 when he was 16 years old. His father then picked out an attractive young woman from a good family, who could provide excellent ties to Basel's business community, to be Daniel's bride. Daniel was horrified—the marriage was clearly impossible. He was shy and would have felt overwhelmed and miserable in the company of this socially accomplished young woman.

Having failed in his attempt at arranging Daniel's marriage, Johann then instructed Daniel to prepare for a career in commerce, arranging an apprenticeship so he could learn the basics of business. Although Johann's father had never attempted to arrange plans for his marriage, he had certainly planned to set up Johann for a life in business. In the same way that Johann in his youth had rejected a life in business, his son Daniel also abhorred the plan. Johann failed to see that it was just as inappropriate for his own son Daniel as it had been for himself, regardless of whether it was a good way to earn a living. Daniel was a handsome young man with a charming, quiet wit when he was in the company of friends, but life in the business world would have been unbearable for him.

"Father, I have no interest in a career in business," Daniel protested.

"Did I ask you if that was what you wanted?" his father demanded.

"No," Daniel admitted.

"You will do as I say, young man!" Johann said.

Eventually Johann relented and allowed Daniel to study medicine instead, and nothing more was said about the suggested marriage. Daniel studied medicine first in Basel and then in Heidelberg, at that time part of the German Palatinate. His doctoral dissertation, which he completed in 1721 at the age of 21, concerned the mechanics of respiration from a mathematical viewpoint. When he had completed that, he applied unsuccessfully for a professorship in anatomy and botany at Basel and again the next year for the professorship in logic.

When it was clear that there was no position for Daniel at the university in Basel, his father Johann arranged for him to travel to Venice to study practical medicine with Pietro Antonio Michelotti, one of the most highly respected physicians in Europe, whose investigations into the way in which blood flows in the human body fascinated Daniel. Michelotti and Daniel worked most congenially together, sharing a love not only for medicine but also, unbeknownst to Daniel's father, for mathematics. Daniel had been so successful in helping Michelotti in a dispute with another Italian physician named Ricatti that Michelotti was delighted to help Daniel in his career, encouraging him to collaborate with him openly in his work both in the hospital and with his private patients.

Johann had also planned for Daniel to study with G. B. Morgagni in Padua, but serious illness forced Daniel to abandon that plan. After several weeks of feverish misery, Daniel finally limped back to health, exercising his mind with mathematics as he began to regain his strength.

Before Daniel left Venice, his mentor Michelotti and the Bernoullis' family friend Christian Goldbach helped Daniel publish his first book, *Exercitationes mathematicae* [*Mathematical Exercises*], a work of serious mathematics. That book allowed Daniel to launch his scientific career. Michelotti and Daniel celebrated happily when the book passed the censors' examination—a major hurdle in Italy at the time—and was actually printed.

With encouragement from Michelotti, Daniel also entered an essay in the Paris Prize competition—the equivalent of the Nobel Prize or the Fields Medal today—the ultimate competition for any scientist at the time. Although he was much too young and inexperienced to expect to win, it was still possible for a novice like Daniel, since every entry was made under a pseudonym, which kept the identity of the entrant sealed until all entries had been judged and allowed each entry to be judged on its merits. Daniel remained in contact with Michelotti for many years, discussing Daniel's mathematical and physical discoveries by letter. Each had profound respect for the other's abilities and insights. Their warm friendship may have been the closest that Daniel ever came to a constructive father–son relationship.

As a result of the publication of his book, Daniel was offered the position of president of a new scientific academy that was about to be established in Genoa, Italy, but he declined that offer. Daniel was uneasy with the political situation in Italy, given the tradition of censoring any scientific work that was seen as contrary to the teachings of the Catholic Church. Galileo had suffered from such censoring, and it appeared that little had changed since his time. Besides that, Daniel was eager to return to his homeland, where he knew his writings would never be subjected to anything like the Italian inquisition. As with many members of the Bernoulli family, Daniel's desire to live in Basel was strong as well.

19
Daniel Bernoulli, the Paris Prize, and the Longitude Problem

In the eighteenth century with its rapidly expanding international trade, technology that would allow a captain to determine the precise location of his ship at sea was an urgent challenge. Since navigation was imprecise at best, shipwrecks were a calamity that occurred far too often, as ships suddenly ran into rocky shores when they thought they were far from any land. Sailors had been able for many years to find their ship's latitude—how far north or south they were—by finding the angle of the sun at its highest point at local noon. However, determining their longitude—the distance east or west—was still a matter of guesswork. Not knowing the ship's longitude meant that the ship's location could be anywhere on a horizontal line stretching around the globe.

If someone could devise a method for knowing the precise time of day or night in a ship at sea, that would allow the sailor to calculate his longitude so that he then could pinpoint his location in the vast ocean. Although Huygens' pendulum clocks were fairly precise on land, they required a steady base and were useless on a ship tossing about for many weeks on the vast ocean. Unfortunately, developing a method to determine the exact time at sea was proving to be extremely difficult. An error of only one minute in 24 hours produced an error of 15 nautical miles or 15 minutes in latitude. During a journey of several months, those many minutes could be compounded to produce a fatal error. The Paris Prize hoped to encourage the scientists of Europe to solve the complex problem.

In 1725, 25-year-old Daniel Bernoulli learned that he had won the prestigious Paris Prize. He received 2500 *livres* [pounds] for his essay "On the Perfection of the Hourglass on a Ship at Sea," in which he described attaching an hourglass to a piece of metal floating in a bowl of mercury, thus minimizing the disturbance to the hourglass in a storm at sea. An hourglass seems to us in the twenty-first century like a primitive tool, but an hourglass can be carefully calibrated, allowing one grain of sand at a time to slip through the opening. Daniel's solution was one step in the desperate eighteenth century search for a method of determining longitude at sea. Although his hourglass was not part of the eventual solution, it was a step that appeared to offer some hope in the quest.

In 1747, at the age of 47, Daniel won the Paris Prize again with his submission of another work on the longitude challenge, sharing it this time with another entrant. This time Daniel's device was a method of controlling the vibrations of a combination of pendulums whose vibrations seemed to cancel each other out so that the result allowed him to power a reasonably precise clock that was only minimally affected by the tossing of the ship at sea. Daniel admitted that his results with the interacting pendulums were surprising, but they seemed to work, and the judges in Paris concurred. Since his solution required several intricate devices working together, however, it was still not the ideal solution. Nevertheless, it was another step in the eventual solution of the longitude problem.

Between 1730 and 1773, John Harrison—an English carpenter and clock maker with little education but with a happy combination of ingenuity and skill—tried another approach. He worked diligently at perfecting one chronometer (a precise clock) after another. His first devices were clocks made of a variety of woods, some of them self-lubricating—an important feature since the quality of lubricating oils available at the time was unreliable. Then he moved on to constructions with a combination of wood and some metal fittings, and then finally to metal alone. Harrison reluctantly had to abandon the use of wood as he perfected his use of fine metal gears and springs. His final result "H4" was a precise timepiece in the form of a watch that was both portable (it would fit in a man's pocket) and accurate to within 1/3 of a second in 24 hours. Since it could be depended on to lose (or gain) the same amount of time each day, that error could be corrected through careful calculation.

Whereas Daniel Bernoulli had written two essays as he worked on the longitude problem at the same time that he was working on many other challenges, Harrison devoted his entire working life of 60 years to the development of his devices. The final product was a wondrous and beautiful machine.

Harrison claimed the reward of £8,750—a phenomenal sum—from the British government. It was Harrison who found the desired solution to the problem on which Daniel had worked on and off, and it is Harrison who deserves the credit for saving untold lives and ships at sea. All four of his chronometers are now on display at the Royal Observatory in Greenwich, England. Alas, none of Daniel's models have survived.